Perturbations of infinitely divisible random fields

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Abstract

Poissonian infinitely divisible processes typically do not have deterministic admissible translations. However they possess rich classes of admissible random translations, called also perturbations, where some of those can be made nearly deterministic. We will discuss constructions of perturbations, transfer of regularity and other relations between infinitely divisible random fields and families of their perturbations.

Outline

1. Introduction
2. Dynkin's isomorphism
3. Perturbation identities for infinitely divisible process

1. Introduction

Notation:

X, Y r.v. in $\mathbb{R}^n$. Relations between the laws $\mathcal{L}(X)$ and $\mathcal{L}(Y)$ can be written as isomorphism identity:

$$\mathbb{E}f(X) = \mathbb{E}[f(Y)\,\eta] \quad \forall f : \mathbb{R}^n \mapsto \mathbb{R}_+$$

for some $\eta \geq 0$ with $\mathbb{E}\eta = 1$.

E.g. $X \overset{d}{=} Y \iff \eta = 1$
$\mathcal{L}(X) \ll \mathcal{L}(Y) \iff \eta \geq 0$
$\mathcal{L}(X) \sim \mathcal{L}(Y) \iff \eta \in (0, \infty) \iff X_{|\mathbb{P}} \overset{d}{=} Y_{|\eta \cdot \mathbb{P}}$

Similar notation can be used for stochastic processes

Example:

Let $(B_t)_{t\in[0,1]}$ be a standard Brownian motion. The Cameron-Martin Formula says that for every absolutely continuous $\varphi : [0,1] \mapsto \mathbb{R}$ with $\varphi' \in L^2$

$$\mathbb{E}F((B_t + \varphi(t))_{t\in[0,1]}) = \mathbb{E}[F((B_t)_{t\in[0,1]})\,\eta] \qquad (1)$$

for all $F : \mathbb{R}^{[0,1]} \mapsto \mathbb{R}$, where

$$\eta = \exp\{\int_0^1 \varphi'(t)dB_t - \frac{1}{2}\int_0^1 |\varphi'(t)|^2\,dt\}.$$

Cameron-Martin Formula:

Given a centered Gaussian process $G = (G_t)_{t \in T}$ over an arbitrary set T and a random variable ξ in L^2_G, the L^2-closure of the subspace spanned by G, we have for any measurable functional $F : \mathbb{R}^T \mapsto \mathbb{R}$

$$\mathbb{E}\left[F\left((G_t + \varphi(t))_{t \in T}\right)\right] = \mathbb{E}\left[F\left((G_t)_{t \in T}\right) e^{\xi - \frac{1}{2}\mathbb{E}\xi^2}\right]$$

where $\varphi(t) = \mathbb{E}(\xi G_t)$.

It is known that (1) does not extend to the Poissonian case.

Counterexample:

If $Y = (Y_t)_{t\in[0,1]}$ is a Poisson process with unit rate, then there is no function $\psi : [0,1] \to \mathbb{R}$, $\psi \not\equiv 0$, such that

$$\mathbb{E}\left[F\left((Y_t + \psi(t))_{t\in[0,1]}\right)\right] = \mathbb{E}\left[F\left((Y_t)_{t\in[0,1]}\right)\eta\right]$$

for all measurable functionals $F : \mathbb{R}^{[0,1]} \mapsto \mathbb{R}$ and some random variable $\eta \geq 0$ with $\mathbb{E}\eta = 1$.

There may be no admissible deterministic translations a for Poissonian infinitely divisible process.
Therefore, we will be searching for random ones, called also perturbations.

Example v-s counterexample:

Let $Y = (Y_t)_{t\in[0,1]}$ is a Poisson process with unit rate. Let ζ be a r.v. in $[0,1]$ with density h and independent of Y. Then

$$\mathbb{E}F\left((Y_t + \mathbf{1}_{[\zeta,1]}(t))_{t\in[0,1]}\right) = \mathbb{E}\left[F\left((Y_t)_{t\in[0,1]}\right)\eta\right],$$

where $\eta = \int_0^1 h(t)dY_t$.

What kind of functionals F can be of interest? A few examples:

- $F\left((Y_t)_{t\in T}\right) = f(Y_{t_1}, \ldots, Y_{t_n})$ cylindrical functional;
- $F\left((Y_t)_{t\in T}\right) = \sup_{t\in T} Y_t$ extremum;
- $F\left((Y_t)_{t\in T}\right) = \int_T |Y_t|^p \, \mu(dt)$ path integral;
- $F\left((Y_t)_{t\in[0,u]}\right) = \int_0^u \delta_y(Y_t)\, dt$ local time;
- $F\left((Y_t)_{t\in[a,b]}\right) = \|Y\|_{BV_p}$ norm in the space of bounded p-variation functions, $p \geq 1$;

2.A. Abstract form of Dynkin's isomorphism (N. Eisenbaum 2008)

Lemma (N. Eisenbaum)

Let $Y = (Y_t)_{t\in T}$ be a nonnegative process with $\theta(t) = \mathbb{E}Y_t < \infty$ for every $t \in T$. Then Y is an infinitely divisible process if and only if for every $s \in T$ having $\theta(s) > 0$, there exists a stochastic process $Z^s = (Z^s_t)_{t\in T}$ independent of Y such that for any measurable functional $F : \mathbb{R}^T \mapsto \mathbb{R}$

$$\mathbb{E}\big[F((Y_t + Z^s_t)_{t\in T})\big] = \mathbb{E}\big[F((Y)_{t\in T}) \cdot \theta(s)^{-1} Y_s\big].$$

Finite dimensional version (characterization of infinite divisibility)

Lemma

Let $Y = (Y_1, \dots, Y_n)$ be a vector of nonnegative random variables with $\theta_i = \mathbb{E}(Y_i) \in (0, \infty)$. The following are equivalent:

(i) *Y is infinitely divisible;*

(ii) *For every $k \leq n$ there exists a vector of nonnegative random variables $Z^k = (Z_1^k, \dots, Z_n^k)$ independent of Y such that for any bounded measurable functional $F : \mathbb{R}^n \mapsto \mathbb{R}$*

$$(Y + Z^k)_{|\mathbb{P}} \stackrel{d}{=} Y_{\ |\theta_k^{-1} Y_k \cdot \mathbb{P}}$$

Proposition (R)

Let $Y = (Y_t)_{t\in T}$ be a nonnegative infinitely divisible process with $\theta(t) = \mathbb{E}Y_t < \infty$ for every $t \in T$. Suppose that for every $s \in T$ having $\theta(s) > 0$, there exists a stochastic process $Z^s = (Z^s_t)_{t\in T}$ independent of Y such that for any measurable functional $F : \mathbb{R}^T \mapsto \mathbb{R}$

$$\mathbb{E}\big[F((Y_t + Z^s_t)_{t\in T})\big] = \mathbb{E}\big[F((Y)_{t\in T}) \cdot \theta(s)^{-1} Y_s\big].$$

If Y is separable in probability with a separant $T_0 = (s_k)_{k\geq 1}$, then the Lévy measure ν of Y is of the form $\nu = \sum_{k\geq 1} \nu_k$, where ν_k are concentrated on disjoint sets $A_k = \{y \in \mathbb{R}^T_+ : y(s_i) = 0\, \forall i < k,\ y(s_k) > 0\}$ and are given by

$$\nu_k(dy) := \theta(s_k)\,\mathbf{1}_{A_k}(y)\,y(s_k)^{-1}\,\mathcal{L}(Z^{s_k})(dy)\,,$$

The drift of Y is given by $c = (\theta(t)\mathbb{P}(Z^t_t = 0))_{t\in T}$.

2.B. Dynkin's isomorphism

A positive real-valued stochastic process $Y = (Y_x)_{x \in E}$ over a set E is called a α-permanental process with kernel $(u(x,y) : x, y \in E)$ if for every $x_1, \ldots, x_n \in E$ and $s_1, \ldots, s_n \geq 0$

$$\mathbb{E} \exp\Big\{ -\sum_{j=1}^{n} s_j Y_{x_j} \Big\} = |I + US|^{-\alpha} \qquad (2)$$

where $U = (u(x_i, x_j) : 1 \leq i, j \leq n)$ and $S = \mathrm{diag}(\mathrm{s}_1, \ldots, \mathrm{s}_\mathrm{n})$ are $n \times n$-matrices, and $\alpha > 0$.

Hence, Y_x's are gamma distributed with shape parameter α and mean $\alpha u(x,x)$ and jointly they have a multivariate multivariate gamma distribution, as defined by (2).

A prototype of a permanental process is a squared Gaussian processes, where $u(x,y)$ is the Gaussian covariance multiplied by 2 and $\alpha = 1/2$.

To formulate the Dynkin Isomorphism Theorem we need more ingredients. $X = (X_t)_{t \geq 0}$ be a transient Markov process with a state space E and 0-potential density $(u(x, y) : x, y \in E)$ with respect to some reference measure. Eisenbaum and Kaspi (2009) showed that for every $\alpha > 0$ a permanental process (2) with such kernel $u(x, y)$ exists and is infinitely divisible.

Assume that X admits the local time $(L_t^x : x \in E, t \geq 0)$, which is normalized to satisfy $\mathbb{E}_x(L_\infty^y) = u(x, y)$. Fix $s \in E$ with $u(s, s) > 0$, and let $\tilde{\mathbb{P}}_s$ be the probability under which the process X starts at s and is killed at its last visit to s.

Then, for any measurable functional $F : \mathbb{R}^E \mapsto \mathbb{R}$,

$$\mathbb{E}\tilde{\mathbb{E}}_s \left[F\left((Y_x + L_\infty^x)_{x\in E}\right)\right] = \mathbb{E}\left[F\left((Y_x)_{x\in E}\right) \frac{Y_s}{\alpha u(s,s)}\right].$$

This identity is known as the Dynkin Isomorphism Theorem. It relates the gamma field to the occupation field.

The identity also enables to transfer path properties of (Y_x), which is easier to handle, to (L_∞^x).

What is a general picture for infinitely divisible processes?

Theorem (R)

Let $X = (X_t)_{t\in T}$ be an infinitely divisible process having a σ-finite Lévy measure ν. Let $Z = (Z_t)_{t\in T}$ be a process independent of X such that $\mathcal{L}(Z) \ll \nu$. Then $\mathcal{L}(X+Z) \ll \mathcal{L}(X)$. Hence, there exists a measurable functional $g : \mathbb{R}^T \mapsto \mathbb{R}_+$ such that for any measurable functional $F : \mathbb{R}^T \mapsto \mathbb{R}$

$$\mathbb{E}F\left((X_t + Z_t)_{t\in T}\right) = \mathbb{E}\left[F\left((X_t)_{t\in T}\right) \cdot g(X)\right] . \tag{3}$$

Remark

Dynkin's isomorphism is in the framework of this theorem. It can be verified that the law of $(L_\infty^x)_{x\in E}$ *under* $\mathbb{P}^s$ *is absolutely continuous with respect to the Lévy measure of the permanental process* $(Y_x)_{x\in E}$. *Using the next theorem one computes that* $g(Y) = Y_s/\mathbb{E}Y_s$.

Theorem (R)

Let $X = (X_t)_{t\in T}$ be an infinitely divisible process of the form $X = G + Y$, where $G = (G_t)_{t\in T}$ is a centered Gaussian process independent of a Poissonian process $Y = (Y_t)_{t\in T}$ having a σ-finite Lévy measure ν and given by its spectral representation

$$Y_t = \int_{\mathbb{R}^T} x(t)[N(dx) - \mathbf{1}_{\{|x(t)|\le 1\}}\nu(dx)] + b(t), \quad t \in T,$$

where N is a Poisson random measure with intensity ν.

Let $Z = (Z_t)_{t\in T}$ be an arbitrary process independent of N and G such that $\mathcal{L}(Z) \ll \nu$ on $\mathcal{B}^T$. Put $q := \frac{d\mathcal{L}(Z)}{d\nu}$ and

$$N(q) = \int_{\mathbb{R}^T} q(x)\, N(dx)\,.$$

Then for any measurable functional $F : \mathbb{R}^T \mapsto \mathbb{R}$

$$\mathbb{E}F\left((X_t + Z_t)_{t\in T}\right) = \mathbb{E}\left[F\left((X_t)_{t\in T}\right) \cdot N(q)\right] \tag{4}$$

Theorem (continue)

Conversely, for any F as above,

$$\mathbb{E}\left[F\left((X_t)_{t\in T}\right)\mathbf{1}_{\{N(q)>0\}}\right] = \mathbb{E}\left[F\left((X_t+Z_t)_{t\in T}\right)\left(N(q)+q(Z)\right)^{-1}\right]$$

where $q(Z) = q\left((Z_t)_{t\in T}\right)$.

Therefore, the distributions $\mathcal{L}(X+Z)$ and $\mathcal{L}(X)$ are equivalent provided $\nu\{x : q(x) > 0\} = \infty$.

Remark

There are two basic directions of applying (3)-(4). *The first one is to start with a process* $Z = (Z_t)_{t \in T}$ *of interest, associate with it (possibly) easier to handle infinitely divisible process* $X = (X_t)_{t \in T}$ *as above, and transfer certain properties of* X *to* Z *via tranfer of regularity property for Lévy measures.*

This will work with such properties as path continuity, boundedness, etc. Using Dynkin's Isomorphism Theorem, Marcus and Rosen derived many results for local times of Markov processes, including Lévy processes.

Another direction of applications of (3)-(4) *is much harder, to derive information about* X *by utilizing* Z.

Theorem (Transfer of regularity)

Let $X = (X_t)_{t \in T}$ be an infinitely divisible process with a σ-finite Lévy measure ν. Assume that paths of X lie in a set U that is a standard Borel space for the σ-algebra $\mathcal{U} = \mathcal{B}^T \cap U$ and U an algebraic subgroup of $\mathbb{R}^T$ under addition. Then ν is concentrated on U in the sense that $\nu_*(\mathbb{R}^T \setminus U) = 0$. Therefore, both $\mathcal{L}(X)$ and its Lévy measure ν are carried by U.

Example (Lévy processes)

Let $X = (X_t)_{t\geq 0}$ be a Lévy process such that $\mathbb{E} e^{iuX_t} = e^{tK(u)}$, where

$$K(u) = \int_{\mathbb{R}} (e^{iux} - 1 - iux\mathbf{1}_{\{|x|\leq 1\}})\, \rho(dx) + icu\,.$$

Let $q : \mathbb{R}_+ \times \mathbb{R} \mapsto \mathbb{R}_+$ be such that $\int_{\mathbb{R}_+\times\mathbb{R}} q(r,v)\, dr\rho(dv) = 1$. Then for any measurable functional $F : \mathbb{R}^{[0,\infty)} \mapsto \mathbb{R}$

$$\mathbb{E}\int_{\mathbb{R}_+\times\mathbb{R}} F\left(\left(X_t + \mathbf{1}_{\{r\leq t\}} v\right)_{t\geq 0}\right)\, q(r,v)\, dr\rho(dv) \\ = \mathbb{E}[F\left((X_t)_{t\geq 0}\right)\cdot g(X)]\,,$$

where

$$g(X) = \sum_{\{r>0:\, \Delta X_r \neq 0\}} q(r, \Delta X_r); \qquad \Delta X_r = X_r - X_{r-}\,.$$

Example (Lévy processes, continue)

Conversely,

$$\mathbb{E}[F(\,(X_t)_{t\geq 0}\,)\mathbf{1}_{\{g(X)>0\}}]$$
$$= \int_{\mathbb{R}_+\times\mathbb{R}} \mathbb{E}\Big[F\left(\left(X_t + \mathbf{1}_{\{r\leq t\}}v\right)_{t\geq 0}\right)\cdot(g(X)+q(r,v))^{-1}\Big]\,q(r,v)$$
$$dr\rho(dv)\,.$$

Moreover, $g(X) > 0$ a.s. if $\int_{\mathbb{R}_+\times\mathbb{R}} \mathbf{1}\{q(r,v)>0\}\,dr\rho(dv) = \infty$.

Remark

The above example extends directly to identities for Lévy sheets

$$X_{\mathbf{t}} = M([0,t_1] \times [0,t_d]), \quad \mathbf{t} = (t_1,\ldots,t_d) \in \mathbb{R}_+^d$$

where M is a homogeneous independently scattered random measure on $\mathbb{R}_+^d$ without the Gaussian part. Then

$$\mathbb{E}\int_{\mathbb{R}_+^d \times \mathbb{R}} F\left(\left(X_{\mathbf{t}} + \mathbf{1}_{\{\mathbf{r}\le\mathbf{t}\}} v\right)_{\mathbf{t}\in\mathbb{R}_+^d}\right) q(\mathbf{r},v)\, d\mathbf{r}\rho(dv)$$
$$= \mathbb{E}[F\left((X_{\mathbf{t}})_{\mathbf{t}\in\mathbb{R}_+^d}\right) \cdot g(X)]\,.$$

Thank you!