STATISTICAL INFERENCE FROM PANEL RANDOM-COEFFICIENT AR(1) DATA

VYTAUTĖ PILIPAUSKAITĖ
(Vilnius University, Lithuania & University of Nantes, France)

Joint with REMIGIJUS LEIPUS (Vilnius),
ANNE PHILIPPE (Nantes) & DONATAS SURGAILIS (Vilnius)

2nd Conference on Ambit Fields and Related Topics
Aarhus, 14–16th August, 2017

OUTLINE

1. INTRODUCTION
2. ASYMPTOTICS OF THE EMPIRICAL C.D.F.
3. GOODNESS-OF-FIT TESTING
4. SIMULATIONS
5. OTHER RESULTS

1. INTRODUCTION

RANDOM-COEFFICIENT AR(1) PROCESS [RCAR(1)]

$$X(t) = aX(t-1) + \zeta(t), \quad t \in \mathbb{Z}, \tag{1}$$

where

- i.i.d. innovations $\{\zeta(t),\ t \in \mathbb{Z}\}$, $\mathrm{E}\zeta(t) = 0$, $\mathrm{E}\zeta^2(t) = 1$,
- random coefficient $a \in [0,1)$ with $\mathrm{E}(1-a^2)^{-1} < \infty$, independent of $\{\zeta(t),\ t \in \mathbb{Z}\}$.

1. INTRODUCTION

RANDOM-COEFFICIENT AR(1) PROCESS [RCAR(1)]

$$X(t) = aX(t-1) + \zeta(t), \quad t \in \mathbb{Z}, \tag{1}$$

where

- i.i.d. innovations $\{\zeta(t),\ t \in \mathbb{Z}\}$, $\mathrm{E}\zeta(t) = 0$, $\mathrm{E}\zeta^2(t) = 1$,
- random coefficient $a \in [0, 1)$ with $\mathrm{E}(1-a^2)^{-1} < \infty$, independent of $\{\zeta(t),\ t \in \mathbb{Z}\}$.

Stationary solution of (1) is given by

$$X(t) = \sum_{s \le t} a^{t-s}\zeta(s), \quad t \in \mathbb{Z},$$

with

$$\mathrm{E}X(t) = 0, \quad \mathrm{E}X(0)X(t) = \mathrm{E}\Big(\frac{a^{|t|}}{1-a^2}\Big) < \infty.$$

Motivation: explanation of long memory in macroeconomic time series (Robinson 1978, Granger 1980, Zaffaroni 2004, Puplinskaitė, Surgailis 2010).

AGGREGATION of independent copies $X_1, \dots, X_N$ of RCAR(1):

$$N^{-1/2} \sum_{i=1}^{N} X_i(t) \to_{\mathrm{fdd}} \mathcal{X}(t), \quad N \to \infty,$$

where $\mathcal{X} := \{\mathcal{X}(t),\ t \in \mathbb{Z}\}$ (= the limit aggregated process) is Gaussian with zero mean and

$$r(t) := \mathrm{E}\mathcal{X}(0)\mathcal{X}(t) = \mathrm{E}X(0)X(t) = \mathrm{E}\Big(\frac{a^{|t|}}{1-a^2}\Big).$$

Motivation: explanation of long memory in macroeconomic time series (Robinson 1978, Granger 1980, Zaffaroni 2004, Puplinskaitė, Surgailis 2010).

AGGREGATION of independent copies $X_1, \ldots, X_N$ of RCAR(1):

$$N^{-1/2} \sum_{i=1}^{N} X_i(t) \to_{\mathrm{fdd}} \mathcal{X}(t), \quad N \to \infty,$$

where $\mathcal{X} := \{\mathcal{X}(t),\ t \in \mathbb{Z}\}$ (= the limit aggregated process) is Gaussian with zero mean and

$$r(t) := \mathrm{E}\mathcal{X}(0)\mathcal{X}(t) = \mathrm{E}X(0)X(t) = \mathrm{E}\Big(\frac{a^{|t|}}{1-a^2}\Big).$$

Assume the AR coefficient a has a density satisfying

$$g(x) \sim g_1(1-x)^{\beta-1}, \quad x \to 1, \quad \text{for some } \beta \in (1,2),\ g_1 > 0. \qquad (2)$$

Motivation: explanation of long memory in macroeconomic time series (Robinson 1978, Granger 1980, Zaffaroni 2004, Puplinskaitė, Surgailis 2010).

AGGREGATION of independent copies $X_1, \dots, X_N$ of RCAR(1):

$$N^{-1/2} \sum_{i=1}^{N} X_i(t) \to_{\mathrm{fdd}} \mathcal{X}(t), \quad N \to \infty,$$

where $\mathcal{X} := \{\mathcal{X}(t),\ t \in \mathbb{Z}\}$ (= the limit aggregated process) is Gaussian with zero mean and

$$r(t) := \mathrm{E}\mathcal{X}(0)\mathcal{X}(t) = \mathrm{E}X(0)X(t) = \mathrm{E}\Big(\frac{a^{|t|}}{1-a^2}\Big).$$

Assume the AR coefficient a has a density satisfying

$$g(x) \sim g_1(1-x)^{\beta-1}, \quad x \to 1, \quad \text{for some } \beta \in (1,2),\ g_1 > 0. \qquad (2)$$

Then $\mathcal{X}$ has LONG MEMORY:

$$r(t) \sim \mathrm{const}\, t^{1-\beta}, \quad t \to \infty, \implies \sum_{t=-\infty}^{\infty} |r(t)| = \infty.$$

Pilipauskaitė, Surgailis 2014:

Let $X_1, \dots, X_N$ be independent copies of RCAR(1) under (2) and

$$S_{N,n}(\tau) := \sum_{i=1}^{N} \sum_{t=1}^{[n\tau]} X_i(t), \quad \tau \geq 0.$$

Pilipauskaitė, Surgailis 2014:

Let $X_1, \dots, X_N$ be independent copies of RCAR(1) under (2) and

$$S_{N,n}(\tau) := \sum_{i=1}^{N} \sum_{t=1}^{[n\tau]} X_i(t), \quad \tau \geq 0.$$

- Let $\beta \in (1,2)$. As $N, n \to \infty$ so that $N/n^\beta \to \mu \in [0,\infty]$,

$$\begin{aligned} N^{-1/2} n^{-H} S_{N,n}(\tau) &\to_{\mathrm{fdd}} \sigma_\infty B_H(\tau) && \text{if } \mu = \infty, \\ N^{-1/\beta} n^{-1/2} S_{N,n}(\tau) &\to_{\mathrm{fdd}} W^{1/2} B(\tau), && \text{if } \mu = 0, \\ N^{-1/\beta} n^{-1/2} S_{N,n}(\tau) &\to_{\mathrm{fdd}} \mu^{1/2} Z(\tau/\mu), && \text{if } \mu \in (0,\infty), \end{aligned}$$

  where B_H is a standard fractional Brownian motion, $H \in (\frac{1}{2}, 1)$,
  B is a standard Brownian motion, $W =_{\mathrm{d}} \mathcal{S}_{\beta/2}(\sigma_0, 1, 0)$,
  Z has a Poisson stochastic integral representation.

Pilipauskaitė, Surgailis 2014:

Let $X_1, \ldots, X_N$ be independent copies of RCAR(1) under (2) and

$$S_{N,n}(\tau) := \sum_{i=1}^{N} \sum_{t=1}^{[n\tau]} X_i(t), \quad \tau \geq 0.$$

- Let $\beta \in (1,2)$. As $N, n \to \infty$ so that $N/n^\beta \to \mu \in [0, \infty]$,

$$\begin{aligned} N^{-1/2} n^{-H} S_{N,n}(\tau) &\to_{\mathrm{fdd}} \sigma_\infty B_H(\tau) && \text{if } \mu = \infty, \\ N^{-1/\beta} n^{-1/2} S_{N,n}(\tau) &\to_{\mathrm{fdd}} W^{1/2} B(\tau), && \text{if } \mu = 0, \\ N^{-1/\beta} n^{-1/2} S_{N,n}(\tau) &\to_{\mathrm{fdd}} \mu^{1/2} Z(\tau/\mu), && \text{if } \mu \in (0, \infty), \end{aligned}$$

  where B_H is a standard fractional Brownian motion, $H \in (\frac{1}{2}, 1)$, B is a standard Brownian motion, $W =_{\mathrm{d}} \mathcal{S}_{\beta/2}(\sigma_0, 1, 0)$, Z has a Poisson stochastic integral representation.

- Let $\beta > 2$. As $N, n \to \infty$ in arbitrary way,

$$N^{-1/2} n^{-1/2} S_{N,n}(\tau) \to_{\mathrm{fdd}} \sigma B(\tau).$$

Pilipauskaitė, Surgailis 2014:

Let $X_1, \ldots, X_N$ be independent copies of RCAR(1) under (2) and

$$S_{N,n}(\tau) := \sum_{i=1}^{N} \sum_{t=1}^{[n\tau]} X_i(t), \quad \tau \geq 0.$$

- Let $\beta \in (1,2)$. As $N, n \to \infty$ so that $N/n^\beta \to \mu \in [0,\infty]$,

$$\begin{aligned} N^{-1/2} n^{-H} S_{N,n}(\tau) &\to_{\mathrm{fdd}} \sigma_\infty B_H(\tau) && \text{if } \mu = \infty, \\ N^{-1/\beta} n^{-1/2} S_{N,n}(\tau) &\to_{\mathrm{fdd}} W^{1/2} B(\tau), && \text{if } \mu = 0, \\ N^{-1/\beta} n^{-1/2} S_{N,n}(\tau) &\to_{\mathrm{fdd}} \mu^{1/2} Z(\tau/\mu), && \text{if } \mu \in (0,\infty), \end{aligned}$$

  where B_H is a standard fractional Brownian motion, $H \in (\frac{1}{2}, 1)$, B is a standard Brownian motion, $W =_{\mathrm{d}} \mathcal{S}_{\beta/2}(\sigma_0, 1, 0)$, Z has a Poisson stochastic integral representation.

- Let $\beta > 2$. As $N, n \to \infty$ in arbitrary way,

$$N^{-1/2} n^{-1/2} S_{N,n}(\tau) \to_{\mathrm{fdd}} \sigma B(\tau).$$

- Related results for network traffic models:
  Mikosch et al. 2002, Gaigalas, Kaj 2003, Kaj, Taqqu 2008, Dombry, Kaj 2011.

PROBLEM
ESTIMATION of the c.d.f. of the AR coefficient

$$G(x) = \mathrm{P}(a \le x), \quad x \in [-1, 1],$$

- from the (limit) aggregated sample:
  Horváth, Leipus 2009, Chong 2006, Leipus et al. 2006, Celov et al. 2010;
- from PANEL RCAR(1) DATA $\{X_i(1), \ldots, X_i(n)\}$, $i = 1, \ldots, N$;

PROBLEM

ESTIMATION of the c.d.f. of the AR coefficient

$$G(x) = \mathrm{P}(a \le x), \quad x \in [-1, 1],$$

- from the (limit) aggregated sample:
  Horváth, Leipus 2009, Chong 2006, Leipus et al. 2006, Celov et al. 2010;
- from PANEL RCAR(1) DATA $\{X_i(1), \ldots, X_i(n)\}$, $i = 1, \ldots, N$;
  - parametric: Robinson 1978, Beran et al. 2010;
  - NONPARAMETRIC: by the empirical c.d.f.

$$\hat{G}_N(x) := \frac{1}{N} \sum_{i=1}^{N} \mathbf{1}(\hat{a}_i \le x), \quad x \in [-1, 1],$$

PROBLEM
ESTIMATION of the c.d.f. of the AR coefficient

$$G(x) = \mathrm{P}(a \le x), \quad x \in [-1, 1],$$

- from the (limit) aggregated sample:
  Horváth, Leipus 2009, Chong 2006, Leipus et al. 2006, Celov et al. 2010;
- from PANEL RCAR(1) DATA $\{X_i(1), \dots, X_i(n)\}$, $i = 1, \dots, N$;
  - parametric: Robinson 1978, Beran et al. 2010;
  - NONPARAMETRIC: by the empirical c.d.f.

$$\hat{G}_N(x) := \frac{1}{N} \sum_{i=1}^{N} \mathbf{1}(\hat{a}_i \le x), \quad x \in [-1, 1],$$

of sample lag 1 autocorrelations (=the estimates of *unobservable* a_i)

$$\hat{a}_i := \frac{\sum_{t=1}^{n-1} (X_i(t) - \bar{X}_i)(X_i(t+1) - \bar{X}_i)}{\sum_{t=1}^{n} (X_i(t) - \bar{X}_i)^2}, \quad \text{where } \bar{X}_i := \frac{1}{n} \sum_{t=1}^{n} X_i(t).$$

2. ASYMPTOTICS OF THE EMPIRICAL C.D.F.

PANEL RCAR(1) DATA MODEL

$\{X_i(t),\ t \in \mathbb{Z}\}$, $i = 1, 2, \ldots$: stationary solutions of

$$\begin{aligned} X_i(t) &= a_i X_i(t-1) + \zeta_i(t), \quad t \in \mathbb{Z}, \\ \zeta_i(t) &= b_i \eta(t) + c_i \xi_i(t), \quad t \in \mathbb{Z}, \end{aligned}$$

2. ASYMPTOTICS OF THE EMPIRICAL C.D.F.

PANEL RCAR(1) DATA MODEL

$\{X_i(t),\ t \in \mathbb{Z}\}$, $i = 1, 2, \ldots$: stationary solutions of

$$\begin{aligned} X_i(t) &= a_i X_i(t-1) + \zeta_i(t), \quad t \in \mathbb{Z}, \\ \zeta_i(t) &= b_i \eta(t) + c_i \xi_i(t), \quad t \in \mathbb{Z}, \end{aligned}$$

under the following assumptions for some $p > 1$ and $\varrho \in (0, 1]$:

- A1 $\{\eta(t)\}$ i.i.d., $\mathrm{E}\eta(t) = 0$, $\mathrm{E}|\eta(t)|^{2p} < \infty$;
- A2 $\{\xi_i(t)\}$, $i = 1, 2, \ldots$, i.i.d., $\mathrm{E}\xi_i(t) = 0$, $\mathrm{E}|\xi_i(t)|^{2p} < \infty$;

2. ASYMPTOTICS OF THE EMPIRICAL C.D.F.

PANEL RCAR(1) DATA MODEL

$\{X_i(t),\ t \in \mathbb{Z}\}$, $i = 1, 2, \ldots$: stationary solutions of

$$\begin{aligned} X_i(t) &= a_i X_i(t-1) + \zeta_i(t), \quad t \in \mathbb{Z}, \\ \zeta_i(t) &= b_i \eta(t) + c_i \xi_i(t), \quad t \in \mathbb{Z}, \end{aligned}$$

under the following assumptions for some $p > 1$ and $\varrho \in (0, 1]$:

- A1 $\{\eta(t)\}$ i.i.d., $\mathrm{E}\eta(t) = 0$, $\mathrm{E}|\eta(t)|^{2p} < \infty$;
- A2 $\{\xi_i(t)\}$, $i = 1, 2, \ldots$, i.i.d., $\mathrm{E}\xi_i(t) = 0$, $\mathrm{E}|\xi_i(t)|^{2p} < \infty$;
- A3 $(b_i, c_i)^\top$, $i = 1, 2, \ldots$, i.i.d. random vectors with b_i, c_i: possibly dependent, $\mathrm{P}(b_i^2 + c_i^2 > 0) = 1$, $\mathrm{E}(b_i^2 + c_i^2) < \infty$;
- A4 $a_i \in (-1, 1)$, $i = 1, 2, \ldots$, i.i.d. with a c.d.f. G satisfying $\mathrm{E}(1 - a_i^2)^{-1} < \infty$;

2. ASYMPTOTICS OF THE EMPIRICAL C.D.F.

PANEL RCAR(1) DATA MODEL

$\{X_i(t),\ t \in \mathbb{Z}\}$, $i = 1, 2, \dots$: stationary solutions of

$$\begin{aligned} X_i(t) &= a_i X_i(t-1) + \zeta_i(t), \quad t \in \mathbb{Z}, \\ \zeta_i(t) &= b_i \eta(t) + c_i \xi_i(t), \quad t \in \mathbb{Z}, \end{aligned}$$

under the following assumptions for some $p > 1$ and $\varrho \in (0, 1]$:

- A1 $\{\eta(t)\}$ i.i.d., $\mathrm{E}\eta(t) = 0$, $\mathrm{E}|\eta(t)|^{2p} < \infty$;
- A2 $\{\xi_i(t)\}$, $i = 1, 2, \dots$, i.i.d., $\mathrm{E}\xi_i(t) = 0$, $\mathrm{E}|\xi_i(t)|^{2p} < \infty$;
- A3 $(b_i, c_i)^\top$, $i = 1, 2, \dots$, i.i.d. random vectors with b_i, c_i: possibly dependent, $\mathrm{P}(b_i^2 + c_i^2 > 0) = 1$, $\mathrm{E}(b_i^2 + c_i^2) < \infty$;
- A4 $a_i \in (-1, 1)$, $i = 1, 2, \dots$, i.i.d. with a c.d.f. G satisfying $\mathrm{E}(1 - a_i^2)^{-1} < \infty$;
- A5 $\{\eta(t)\}$, $\{\xi_i(t)\}$, $(b_i, c_i)^\top$, a_i are independent for every $i = 1, 2, \dots$

2. ASYMPTOTICS OF THE EMPIRICAL C.D.F.

PANEL RCAR(1) DATA MODEL

$\{X_i(t),\ t \in \mathbb{Z}\}$, $i = 1, 2, \ldots$: stationary solutions of

$$X_i(t) = a_i X_i(t-1) + \zeta_i(t), \quad t \in \mathbb{Z},$$
$$\zeta_i(t) = b_i \eta(t) + c_i \xi_i(t), \quad t \in \mathbb{Z},$$

under the following assumptions for some $p > 1$ and $\varrho \in (0, 1]$:

- A1 $\{\eta(t)\}$ i.i.d., $\mathrm{E}\eta(t) = 0$, $\mathrm{E}|\eta(t)|^{2p} < \infty$;
- A2 $\{\xi_i(t)\}$, $i = 1, 2, \ldots$, i.i.d., $\mathrm{E}\xi_i(t) = 0$, $\mathrm{E}|\xi_i(t)|^{2p} < \infty$;
- A3 $(b_i, c_i)^\top$, $i = 1, 2, \ldots$, i.i.d. random vectors with b_i, c_i: possibly dependent, $\mathrm{P}(b_i^2 + c_i^2 > 0) = 1$, $\mathrm{E}(b_i^2 + c_i^2) < \infty$;
- A4 $a_i \in (-1, 1)$, $i = 1, 2, \ldots$, i.i.d. with a c.d.f. G satisfying $\mathrm{E}(1 - a_i^2)^{-1} < \infty$;
- A5 $\{\eta(t)\}$, $\{\xi_i(t)\}$, $(b_i, c_i)^\top$, a_i are independent for every $i = 1, 2, \ldots$
- A6 G is ϱ-Hölder continuous: $\exists L > 0$ such that $|G(x) - G(y)| \le L|x - y|^\varrho$, $\forall x, y \in [-1, 1]$;

Fix $i = 1, 2, \ldots$. The sample lag 1 autocorrelation of $\{X_i(1), \ldots, X_i(n)\}$

$$\hat{a}_i = \frac{\sum_{t=1}^{n-1}(X_i(t) - \bar{X}_i)(X_i(t+1) - \bar{X}_i)}{\sum_{t=1}^{n}(X_i(t) - \bar{X}_i)^2}, \quad \text{where } \bar{X}_i = \frac{1}{n}\sum_{t=1}^{n} X_i(t),$$

is invariant to shift and scale transformations of X_i; $|\hat{a}_i| \leq 1$ a.s.

Fix $i = 1, 2, \ldots$. The sample lag 1 autocorrelation of $\{X_i(1), \ldots, X_i(n)\}$

$$\hat{a}_i = \frac{\sum_{t=1}^{n-1}(X_i(t) - \bar{X}_i)(X_i(t+1) - \bar{X}_i)}{\sum_{t=1}^{n}(X_i(t) - \bar{X}_i)^2}, \quad \text{where } \bar{X}_i = \frac{1}{n}\sum_{t=1}^{n} X_i(t),$$

is invariant to shift and scale transformations of X_i; $|\hat{a}_i| \le 1$ a.s.

THEOREM

Assume the panel RCAR(1) data model under A1–A6. If $N, n \to \infty$ so that $N n^{-\frac{2\varrho}{\varrho+p}(\frac{p}{2}\wedge(p-1))} \to 0$, then

$$\sqrt{N}(\hat{G}_N(x) - G(x)) \to_{D[-1,1]} W(x),$$

where $\{W(x),\ x \in [-1,1]\}$ is a Gaussian process with zero mean and $\mathrm{E}\, W(x)\, W(y) = G(x \wedge y) - G(x)G(y)$; and $\to_{D[-1,1]}$ denotes the weak convergence in $D[-1,1]$ with the uniform metric.

Fix $i = 1, 2, \ldots$. The sample lag 1 autocorrelation of $\{X_i(1), \ldots, X_i(n)\}$

$$\hat{a}_i = \frac{\sum_{t=1}^{n-1}(X_i(t) - \bar{X}_i)(X_i(t+1) - \bar{X}_i)}{\sum_{t=1}^{n}(X_i(t) - \bar{X}_i)^2}, \quad \text{where } \bar{X}_i = \frac{1}{n}\sum_{t=1}^{n} X_i(t),$$

is invariant to shift and scale transformations of X_i; $|\hat{a}_i| \le 1$ a.s.

THEOREM

Assume the panel RCAR(1) data model under A1–A6. If $N, n \to \infty$ so that $N n^{-\frac{2\varrho}{\varrho+p}(\frac{p}{2} \wedge (p-1))} \to 0$, then

$$\sqrt{N}(\hat{G}_N(x) - G(x)) \to_{D[-1,1]} W(x),$$

where $\{W(x),\ x \in [-1,1]\}$ is a Gaussian process with zero mean and $\mathrm{E}\, W(x)\, W(y) = G(x \wedge y) - G(x) G(y)$; and $\to_{D[-1,1]}$ denotes the weak convergence in $D[-1,1]$ with the uniform metric.

- Thm. applies to long panels: if $\varrho = 1$, then for very large p, we assume $N/n^{p/(1+p)} \to 0$, where $p/(1+p) \approx 1$.

IDEA OF THE PROOF. It suffices to show that

$$\sup_{x\in[-1,1]} |\hat{D}_N(x)| \to_{\mathrm{P}} 0,$$

where

$$\hat{D}_N(x) := N^{-1/2} \sum_{i=1}^{N} (\mathbf{1}(\hat{a}_i \leq x) - \mathbf{1}(a_i \leq x)).$$

IDEA OF THE PROOF. It suffices to show that

$$\sup_{x \in [-1,1]} |\hat{D}_N(x)| \to_{\mathrm{P}} 0,$$

where

$$\hat{D}_N(x) := N^{-1/2} \sum_{i=1}^{N} (\mathbf{1}(\hat{a}_i \leq x) - \mathbf{1}(a_i \leq x)).$$

For $\varepsilon > 0$, we have

$$|\hat{D}_N(x)| \leq N^{-1/2} \sum_{i=1}^{N} (\mathbf{1}(x - \varepsilon < a_i \leq x + \varepsilon) + \mathbf{1}(|\hat{a}_i - a_i| > \varepsilon)).$$

IDEA OF THE PROOF. It suffices to show that

$$\sup_{x\in[-1,1]} |\hat{D}_N(x)| \to_{\mathrm{P}} 0,$$

where

$$\hat{D}_N(x) := N^{-1/2}\sum_{i=1}^{N}(\mathbf{1}(\hat{a}_i \leq x) - \mathbf{1}(a_i \leq x)).$$

For $\varepsilon > 0$, we have

$$|\hat{D}_N(x)| \leq N^{-1/2}\sum_{i=1}^{N}(\mathbf{1}(x-\varepsilon < a_i \leq x+\varepsilon) + \mathbf{1}(|\hat{a}_i - a_i| > \varepsilon)).$$

Use ϱ-Hölder continuity of G with $\varepsilon^{\varrho+p} \sim n^{-\frac{p}{2}\wedge(p-1)} = o(1)$ and

PROPOSITION 1

Fix $i = 1, 2, \ldots$. Under A1–A5, for any $\varepsilon \in (0,1)$ and $n = 1, 2, \ldots$, it holds

$$\mathrm{P}(|\hat{a}_i - a_i| > \varepsilon) \leq C(n^{-\frac{p}{2}\wedge(p-1)}\varepsilon^{-p} + n^{-1})$$

with $C > 0$ independent of n, ε.

3. GOODNESS-OF-FIT TESTING

SIMPLE GoF

$$H_0 : G = G_0, \quad H_1 : G = G_0,$$

with G_0 completely specified.

3. GOODNESS-OF-FIT TESTING

SIMPLE GoF

$$H_0 : G = G_0, \quad H_1 : G = G_0,$$

with G_0 completely specified.

- The Kolmogorov–Smirnov test rejects H_0 at level $\omega \in (0, 1)$ if

$$\sqrt{N} \sup_x |\hat{G}_N(x) - G_0(x)| > c(\omega),$$

  where $c(\omega)$ is the upper ω-quantile of the Kolmogorov distribution.
- It has asymptotic size ω and is consistent provided the assumptions of Thm. hold.

COMPOSITE GoF

$$H_0 : G \in \mathcal{G} := \{G_\theta,\ \theta \in (1,\infty)^2\}, \quad H_1 : G \notin \mathcal{G},$$

with $\mathcal{G}$ being the family of the beta c.d.f.s parametrized by $\theta = (\alpha, \beta)^\top$:

$$G_\theta(x) = \frac{1}{\mathrm{B}(\alpha,\beta)} \int_0^x t^{\alpha-1}(1-t)^{\beta-1}\mathrm{d}t, \quad x \in [0,1],$$

where $\mathrm{B}(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$ is the beta function.

COMPOSITE GoF

$$H_0 : G \in \mathcal{G} := \{G_\theta,\ \theta \in (1,\infty)^2\}, \quad H_1 : G \notin \mathcal{G},$$

with $\mathcal{G}$ being the family of the beta c.d.f.s parametrized by $\theta = (\alpha, \beta)^\top$:

$$G_\theta(x) = \frac{1}{\mathrm{B}(\alpha,\beta)} \int_0^x t^{\alpha-1}(1-t)^{\beta-1}\mathrm{d}t, \quad x \in [0,1],$$

where $\mathrm{B}(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$ is the beta function.

- The Kolmogorov–Smirnov statistic with estimated parameters $\hat{\theta}_N = (\hat{\alpha}_N, \hat{\beta}_N)^\top$ (by method of moments):

$$\sqrt{N} \sup_x |\hat{G}_N(x) - G_{\hat{\theta}_N}(x)|.$$

Fix $m \in \mathbb{N}$. Let $\mu = (\mu^{(1)}, \dots, \mu^{(m)})^\top$ and $\hat{\mu}_N = (\hat{\mu}_N^{(1)}, \dots, \hat{\mu}_N^{(m)})^\top$, where

$$\mu^{(u)} := \mathrm{E}a^u, \quad \hat{\mu}_N^{(u)} := \frac{1}{N}\sum_{i=1}^{N} \hat{a}_i^u, \quad u = 1, \dots, m.$$

Fix $m \in \mathbb{N}$. Let $\mu = (\mu^{(1)}, \ldots, \mu^{(m)})^\top$ and $\hat{\mu}_N = (\hat{\mu}_N^{(1)}, \ldots, \hat{\mu}_N^{(m)})^\top$, where

$$\mu^{(u)} := \mathrm{E}a^u, \quad \hat{\mu}_N^{(u)} := \frac{1}{N} \sum_{i=1}^{N} \hat{a}_i^u, \quad u = 1, \ldots, m.$$

PROPOSITION

Assume the panel RCAR(1) data model under A1–A5. If $N, n \to \infty$ so that $N n^{-\frac{2}{1+p}(\frac{p}{2} \wedge (p-1))} \to 0$, then

$$\sqrt{N}(\hat{\mu}_N - \mu) \to_{\mathrm{d}} \mathcal{N}(0, \Sigma),$$

where $\Sigma := (\mathrm{cov}(a^u, a^v))_{1 \le u,v \le m}$.

Fix $m \in \mathbb{N}$. Let $\mu = (\mu^{(1)}, \ldots, \mu^{(m)})^\top$ and $\hat{\mu}_N = (\hat{\mu}_N^{(1)}, \ldots, \hat{\mu}_N^{(m)})^\top$, where

$$\mu^{(u)} := \mathrm{E} a^u, \quad \hat{\mu}_N^{(u)} := \frac{1}{N} \sum_{i=1}^{N} \hat{a}_i^u, \quad u = 1, \ldots, m.$$

PROPOSITION

Assume the panel RCAR(1) data model under A1–A5. If $N, n \to \infty$ so that $N n^{-\frac{2}{1+p}(\frac{p}{2} \wedge (p-1))} \to 0$, then

$$\sqrt{N}(\hat{\mu}_N - \mu) \to_{\mathrm{d}} \mathcal{N}(0, \Sigma),$$

where $\Sigma := (\mathrm{cov}(a^u, a^v))_{1 \le u, v \le m}$.

- Robinson 1978: AN of a different estimator of μ for fixed n as $N \to \infty$ if $\{\zeta_i(t) \equiv \xi_i(t)\}$ and $\mathrm{E}(1 - a_i^2)^{-2} < \infty$ (short memory).

The method-of-moments estimator $\hat{\theta}_N = (\hat{\alpha}_N, \hat{\beta}_N)^\top$ of beta parameter θ:

$$\hat{\alpha}_N = \frac{\hat{\mu}_N^{(1)}(\hat{\mu}_N^{(1)} - \hat{\mu}_N^{(2)})}{\hat{\mu}_N^{(2)} - (\hat{\mu}_N^{(1)})^2}, \qquad \hat{\beta}_N = \frac{(1 - \hat{\mu}_N^{(1)})(\hat{\mu}_N^{(1)} - \hat{\mu}_N^{(2)})}{\hat{\mu}_N^{(2)} - (\hat{\mu}_N^{(1)})^2}.$$

The method-of-moments estimator $\hat{\theta}_N = (\hat{\alpha}_N, \hat{\beta}_N)^\top$ of beta parameter θ:

$$\hat{\alpha}_N = \frac{\hat{\mu}_N^{(1)}(\hat{\mu}_N^{(1)} - \hat{\mu}_N^{(2)})}{\hat{\mu}_N^{(2)} - (\hat{\mu}_N^{(1)})^2}, \qquad \hat{\beta}_N = \frac{(1 - \hat{\mu}_N^{(1)})(\hat{\mu}_N^{(1)} - \hat{\mu}_N^{(2)})}{\hat{\mu}_N^{(2)} - (\hat{\mu}_N^{(1)})^2}.$$

COROLLARY

Assume the panel RCAR(1) data model under A1–A6 with $G = G_\theta$, $\theta \in (1, \infty)^2$. If $N, n \to \infty$ so that $Nn^{-\frac{2}{1+p}(\frac{p}{2} \wedge (p-1))} \to 0$, then

$$\sqrt{N}(\hat{\theta}_N - \theta) \to_{\mathrm{d}} \mathcal{N}(0, \Lambda_\theta),$$

where $\Lambda_\theta := \Delta^{-1}\Sigma(\Delta^{-1})^\top$, $\Delta := \partial\mu/\partial\theta$, Σ as in Prop.

The method-of-moments estimator $\hat{\theta}_N = (\hat{\alpha}_N, \hat{\beta}_N)^\top$ of beta parameter θ:

$$\hat{\alpha}_N = \frac{\hat{\mu}_N^{(1)}(\hat{\mu}_N^{(1)} - \hat{\mu}_N^{(2)})}{\hat{\mu}_N^{(2)} - (\hat{\mu}_N^{(1)})^2}, \qquad \hat{\beta}_N = \frac{(1-\hat{\mu}_N^{(1)})(\hat{\mu}_N^{(1)} - \hat{\mu}_N^{(2)})}{\hat{\mu}_N^{(2)} - (\hat{\mu}_N^{(1)})^2}.$$

COROLLARY

Assume the panel RCAR(1) data model under A1–A6 with $G = G_\theta$, $\theta \in (1,\infty)^2$. If $N, n \to \infty$ so that $N n^{-\frac{2}{1+p}(\frac{p}{2}\wedge(p-1))} \to 0$, then

$$\sqrt{N}(\hat{\theta}_N - \theta) \to_{\mathrm{d}} \mathcal{N}(0, \Lambda_\theta),$$

where $\Lambda_\theta := \Delta^{-1}\Sigma(\Delta^{-1})^\top$, $\Delta := \partial\mu/\partial\theta$, Σ as in Prop. Moreover,

$$\sqrt{N}(\hat{G}_N(x) - G_{\hat{\theta}_N}(x)) \to_{D[0,1]} V_\theta(x),$$

where $\{V_\theta(x),\ x \in [0,1]\}$ is a Gaussian process with zero mean and

$$\begin{aligned} \mathrm{E}\, V_\theta(x)\, V_\theta(y) = {} & G_\theta(x\wedge y) - G_\theta(x) G_\theta(y) + \partial_\theta G_\theta(x)^\top \Lambda_\theta \partial_\theta G_\theta(y) \\ & - \int_0^x l_\theta(u)^\top \mathrm{d}\, G_\theta(u) \partial_\theta G_\theta(y) - \int_0^y l_\theta(u)^\top \mathrm{d}\, G_\theta(u) \partial_\theta G_\theta(x) \end{aligned}$$

with $\partial_\theta G_\theta(x) := \partial G_\theta(x)/\partial\theta,\ l_\theta(x) := \Delta^{-1}(x - \mu^{(1)}, x^2 - \mu^{(2)})^\top$.

COMPOSITE GoF

$$H_0 : G \in \mathcal{G} = \{G_\theta,\ \theta \in (1,\infty)^2\}, \quad H_1 : G \notin \mathcal{G}$$

with $\mathcal{G}$ being the family of the beta c.d.f.s parametrized by $\theta = (\alpha, \beta)^\top$.

H_0 is rejected at level $\varpi \in (0,1)$ if

$$\sqrt{N} \sup_x |\hat{G}_N(x) - G_{\hat{\theta}_N}(x)| > c_{\hat{\theta}_N}(\omega),$$

where

$$\mathrm{P}\big(\sup_x |V_\theta(x)| > c_\theta(\omega)\big) = \omega.$$

COMPOSITE GoF

$$H_0 : G \in \mathcal{G} = \{G_\theta,\ \theta \in (1,\infty)^2\}, \quad H_1 : G \notin \mathcal{G}$$

with $\mathcal{G}$ being the family of the beta c.d.f.s parametrized by $\theta = (\alpha, \beta)^\top$.

H_0 is rejected at level $\varpi \in (0,1)$ if

$$\sqrt{N} \sup_x |\hat{G}_N(x) - G_{\hat{\theta}_N}(x)| > c_{\hat{\theta}_N}(\omega),$$

where

$$\mathrm{P}\big(\sup_x |V_\theta(x)| > c_\theta(\omega) \big) = \omega.$$

- The test has asymptotic size ω and is consistent (since $\hat{\mu}_N \to_{\mathrm{P}} \mu$ implies $\hat{\theta}_N \to_{\mathrm{P}} \theta$ and $c_\theta(\omega)$ is continuous in θ) provided the assumptions of Cor. hold.
- Parametric bootstrap can also produce asymptotically correct critical values.

4. SIMULATIONS

Beran et al. 2010:

- Let $X_1, \ldots, X_N$ be independent copies of RCAR(1) with $\zeta_i(t) \equiv \xi_i(t) =_{\mathrm{d}} \mathcal{N}(0,1)$ and

$$\mathrm{P}(a_i^2 \le x) = G_\theta(x), \quad x \in [0,1], \quad \theta \in (1,\infty)^2.$$

4. SIMULATIONS

Beran et al. 2010:

- Let $X_1, \ldots, X_N$ be independent copies of RCAR(1) with $\zeta_i(t) \equiv \xi_i(t) =_{\mathrm{d}} \mathcal{N}(0,1)$ and

$$\mathrm{P}(a_i^2 \le x) = G_\theta(x), \quad x \in [0,1], \quad \theta \in (1,\infty)^2.$$

- $\tilde{\theta}_N$ is defined as a maximum likelihood estimator of θ with unobservable a_i replaced by

$$\tilde{a}_i := \min(\max(\hat{a}_i, \kappa), 1-\kappa), \quad i = 1, \ldots, N,$$

  where $\kappa > 0$ is a truncation parameter.

4. SIMULATIONS

Beran et al. 2010:

- Let $X_1, \ldots, X_N$ be independent copies of RCAR(1) with $\zeta_i(t) \equiv \xi_i(t) =_{\mathrm{d}} \mathcal{N}(0,1)$ and

$$\mathrm{P}(a_i^2 \leq x) = G_\theta(x), \quad x \in [0,1], \quad \theta \in (1,\infty)^2.$$

- $\tilde{\theta}_N$ is defined as a maximum likelihood estimator of θ with unobservable a_i replaced by

$$\tilde{a}_i := \min(\max(\hat{a}_i, \kappa), 1-\kappa), \quad i = 1, \ldots, N,$$

  where $\kappa > 0$ is a truncation parameter.

- If $\sqrt{N}\kappa^{-2}n^{-1} \to 0$, $\sqrt{N}\kappa^{\min(\alpha,\beta)} \to 0$ and $(\log \kappa)^2 N^{-1/2} \to 0$ as $N, n \to \infty$, $\kappa \to 0$, then

$$\sqrt{N}(\tilde{\theta}_N - \theta) \to_{\mathrm{d}} \mathcal{N}(0, A^{-1}(\theta)).$$

Simulation procedure to compare:

$$\begin{aligned} T_{KS} &:= \sqrt{N} \sup_x |\hat{G}_N(x) - G_{\theta_0}(x)|, \\ T_{MLE} &:= N(\tilde{\theta}_N - \theta_0)^\top A(\theta_0)(\tilde{\theta}_N - \theta_0) \end{aligned}$$

in testing

$$H_0 : G = G_{\theta_0} \ (\theta = \theta_0), \quad H_1 : G \neq G_{\theta_0} \ (\theta \neq \theta_0)$$

for $\theta_0 = (2, 1.4)^\top$.

Simulation procedure to compare:

$$
\begin{aligned}
T_{KS} &:= \sqrt{N} \sup_x |\hat{G}_N(x) - G_{\theta_0}(x)|, \\
T_{MLE} &:= N(\tilde{\theta}_N - \theta_0)^\top A(\theta_0)(\tilde{\theta}_N - \theta_0)
\end{aligned}
$$

in testing

$$
H_0 : G = G_{\theta_0} \ (\theta = \theta_0), \quad H_1 : G \neq G_{\theta_0} \ (\theta \neq \theta_0)
$$

for $\theta_0 = (2, 1.4)^\top$.

- The same θ_0, N, n as in Beran et al. 2010.
- $\beta \in (1, 2)$ implies the long memory in RCAR(1).
- p-value $:= 1 - F_i(T_i)$, where $F_i :=$ limit c.d.f. of T_i under H_0, $i = KS, MLE$.
- If the asymptotic size of the test is correct, then the asymptotic distribution of the p-value is uniform on $[0, 1]$.

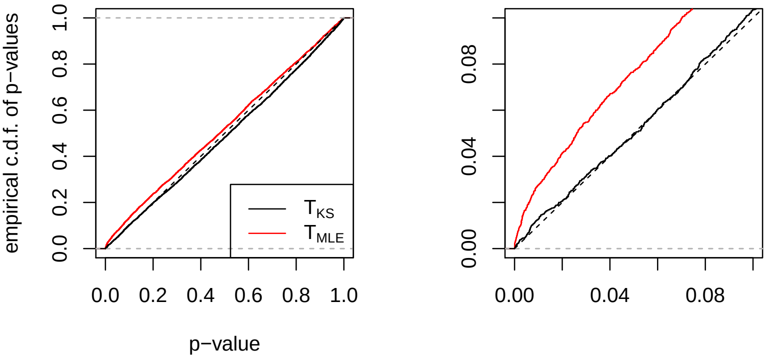

Figure: [left] Empirical c.d.f. of p-values of T_{KS} and T_{MLE} from 5000 replications of a panel with $N = 250$, $n = 817$ under $H_0 : \theta = (2, 1.4)^\top$. [right] Zoom-in on the region of interest: p-values smaller than 0.1.

$\omega = 5\%$

β	1.2	1.3	1.4	1.5	1.6
T_{KS}	.532	.137	.049	.208	.576
T_{MLE}	.500	.104	.077	.313	.735

$\omega = 10\%$

β	1.2	1.3	1.4	1.5	1.6
T_{KS}	.653	.223	.103	.315	.702
T_{MLE}	.634	.184	.134	.421	.827

Table: Empirical probability to reject $H_0 : \theta = (2, 1.4)^\top$ at levels $\omega = 5\%$, 10%. 5000 replications of a panel with $N = 250$, $n = 817$ and $\theta = (2, \beta)^\top$. The column for $\beta = 1.4$ provides the empirical size.

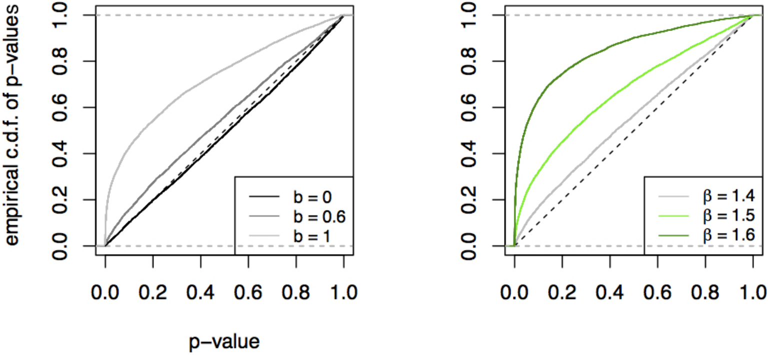

Figure: Empirical c.d.f. of 5000 p-values of T_{KS} for testing $H_0 : \theta = (2, 1.4)^\top$ from a panel comprising $N = 250$ RCAR(1) series of length
[left] $n = 817$ under H_0 and dependence structure $(b_i, c_i)^\top = (b, \sqrt{1 - b^2})^\top$;
[right] $n = 5500$ under $\theta = (2, \beta)^\top$ and $(b_i, c_i)^\top = (1, 0)^\top$, i.e. all series are driven by common innovations.

CONCLUSIONS:

- We do not observe an important loss of the power for T_{KS} compared to T_{MLE}.
- T_{KS} does not require to choose any truncation parameter contrary to T_{MLE}.
- We can use T_{KS} under weaker assumptions on (moments, dependence structure of) RCAR(1) innovations.

5. OTHER RESULTS

Assume

A6' G is continuously differentiable with derivative g.

Its KERNEL DENSITY ESTIMATOR is

$$\hat{g}_N(x) := \frac{1}{Nh}\sum_{i=1}^{N} K\Big(\frac{x-\hat{a}_i}{h}\Big), \quad x \in \mathbb{R},$$

where the kernel $K : [-1,1] \to \mathbb{R}$ is Lipschitz, $K(x) = 0$, $x \in \mathbb{R} \setminus [-1,1]$ and $h > 0$ is a bandwidth.

5. OTHER RESULTS

Assume

A6' G is continuously differentiable with derivative g.

Its KERNEL DENSITY ESTIMATOR is

$$\hat{g}_N(x) := \frac{1}{Nh} \sum_{i=1}^{N} K\Big(\frac{x - \hat{a}_i}{h}\Big), \quad x \in \mathbb{R},$$

where the kernel $K : [-1, 1] \to \mathbb{R}$ is Lipschitz, $K(x) = 0$, $x \in \mathbb{R} \setminus [-1, 1]$ and $h > 0$ is a bandwidth.

Under certain conditions on $N, n \to \infty$, $h \to 0$,

$$\int_{-\infty}^{\infty} \mathrm{E}|\hat{g}_N(x) - g(x)|^2 \mathrm{d}x \to 0 \quad \text{and} \quad \frac{\hat{g}_N(x) - \mathrm{E}\hat{g}_N(x)}{\sqrt{\mathrm{Var}(\hat{g}_N(x))}} \to \mathcal{N}(0, 1).$$

Assume

A6" $\exists \epsilon > 0$ such that G is continuously differentiable on $(1-\epsilon, 1)$ with derivative g satisfying

$$g(x) = g_1(1-x)^{\beta-1}(1 + O((1-x)^{\nu})), \quad x \to 1,$$

for some $\beta > 1$ and $g_1 > 0$.

Assume

A6" $\exists \epsilon > 0$ such that G is continuously differentiable on $(1-\epsilon, 1)$ with derivative g satisfying

$$g(x) = g_1(1-x)^{\beta-1}(1 + O((1-x)^{\nu})), \quad x \to 1,$$

for some $\beta > 1$ and $g_1 > 0$.

Then $Y := 1/(1-a)$ satisfies

$$\mathrm{P}(Y > y) = (g_1/\beta) y^{-\beta}(1 + O(y^{-\nu})), \quad y \to \infty.$$

Assume

A6" $\exists \epsilon > 0$ such that G is continuously differentiable on $(1-\epsilon, 1)$ with derivative g satisfying

$$g(x) = g_1(1-x)^{\beta-1}(1 + O((1-x)^\nu)), \quad x \to 1,$$

for some $\beta > 1$ and $g_1 > 0$.

Then $Y := 1/(1-a)$ satisfies

$$\mathrm{P}(Y > y) = (g_1/\beta) y^{-\beta}(1 + O(y^{-\nu})), \quad y \to \infty.$$

Goldie, Smith 1987:
Let $Y, Y_1, \ldots, Y_N$ be i.i.d. r.v.s.
THE ESTIMATOR OF THE TAIL-INDEX β is given by

$$\beta_N = \frac{\sum_{i=1}^N \mathbf{1}(Y_i \geq y)}{\sum_{i=1}^N \mathbf{1}(Y_i \geq y) \ln(Y_i/y)},$$

where $y > 0$ is a threshold.

Let

$$\tilde{\beta}_N := \frac{\sum_{i=1}^N \mathbf{1}(\hat{a}_i > 1-\delta)}{\sum_{i=1}^N \mathbf{1}(\tilde{a}_i > 1-\delta)\ln(\delta/(1-\tilde{a}_i))},$$

where $\delta > 0$ is a threshold close to 0 and

$$\tilde{a}_i := \min(\hat{a}_i, 1-\delta^2), \quad i = 1, \ldots, N.$$

Let

$$\tilde{\beta}_N := \frac{\sum_{i=1}^N \mathbf{1}(\hat{a}_i > 1-\delta)}{\sum_{i=1}^N \mathbf{1}(\tilde{a}_i > 1-\delta)\ln(\delta/(1-\tilde{a}_i))},$$

where $\delta > 0$ is a threshold close to 0 and

$$\tilde{a}_i := \min(\hat{a}_i, 1-\delta^2), \quad i = 1, \dots, N.$$

THEOREM

Assume the panel RCAR(1) data model under A1–A6 and $N \to \infty$, so that $n \to \infty$, $\delta \to 0$ and $N\delta^{\beta+2(\beta\wedge\nu)} \to 0$, $N\delta^\beta/(\ln\delta)^4 \to \infty$ and

$$\begin{aligned} \sqrt{N\delta^\beta}\gamma\ln\delta &\to 0 \quad \text{if } 1 < p \le 2, \\ \sqrt{N\delta^\beta}((n\delta^\beta)^{-1} \vee \gamma)\ln\delta &\to 0 \quad \text{if } p > 2, \end{aligned}$$

where $\gamma := \gamma_N = (n^{(p-1)\wedge(p/2)}\delta^{p+\beta})^{-1/(p+1)}$. Then

$$\sqrt{\hat{K}_N}(\tilde{\beta}_N - \beta) \to_{\mathrm{d}} \mathcal{N}(0, \beta^2),$$

where $\hat{K}_N := \sum_{i=1}^N \mathbf{1}(\hat{a}_i > 1-\delta)$.

TEST FOR LONG MEMORY:

$$H_0 : \beta \geq 2, \quad H_1 : \beta < 2 \text{ (RCAR(1) has long memory)}$$

H_0 is rejected at level $\omega \in (0, 1)$ if

$$\tilde{T}_N := \sqrt{\hat{K}_N}(\tilde{\beta}_N - 2)/\tilde{\beta}_N < z(\omega),$$

where $z(\omega)$ is the ω-quantile of standard normal distribution.

Under assumptions of Thm.,

- $\tilde{T}_N \to_{\mathrm{d}} \mathcal{N}(0, 1)$ if $\beta = 2$ and $\tilde{T}_N \to_{\mathrm{p}} -\infty$ if $\beta < 2$.

TEST FOR LONG MEMORY:

$$H_0 : \beta \geq 2, \quad H_1 : \beta < 2 \text{ (RCAR(1) has long memory)}$$

H_0 is rejected at level $\omega \in (0,1)$ if

$$\tilde{T}_N := \sqrt{\hat{K}_N}(\tilde{\beta}_N - 2)/\tilde{\beta}_N < z(\omega),$$

where $z(\omega)$ is the ω-quantile of standard normal distribution.

Under assumptions of Thm.,

- $\tilde{T}_N \to_{\mathrm{d}} \mathcal{N}(0,1)$ if $\beta = 2$ and $\tilde{T}_N \to_{\mathrm{p}} -\infty$ if $\beta < 2$.

β	1.5	1.75	2	2.25	2.5	2.75
i.i.d.	0.476	0.189	0.055	0.031	0.025	0.022
$n = 1000$	0.186	0.058	0.025	0.015	0.010	0.014
$n = 5000$	0.368	0.137	0.050	0.031	0.024	0.015
$n = 10000$	0.410	0.130	0.050	0.039	0.028	0.016

Table: Empirical probability to reject $H_0 : \beta \geq 2$ at level $\omega = 5\%$.
The i.i.d. row stands for testing from unobservable AR coefficients.
Three last rows correspond to panel data comprising $N = 1000$ independent RCAR(1) series of length n. The AR coefficient is beta distributed wih parameters $(2, \beta)$. Estimations are made from 1000 independent replications.

REFERENCES

[1] Beran, J., Schützner, M. and Ghosh, S. (2010). From short to long memory: Aggregation and estimation. *Comput. Statist. Data Anal.* **54**, 2432–2442.

[2] Goldie, C.M., Smith, R.L. (1987) Slow variation with remainder: theory and applications. *Quart. J. Math. Oxford* **38**, 45–71.

[3] Granger, C.W.J. (1980). Long memory relationships and the aggregation of dynamic models. *J. Econometrics.* **14**, 227–238.

[4] Leipus, R., Philippe, A., Pilipauskaitė, V. and Surgailis, D. (2017). Nonparametric estimation of the distribution of the autoregressive coefficient from panel random-coefficient AR(1) data. *J. Multivar. Anal.* **153**, 121–135.

[5] Pilipauskaitė, V., Surgailis, D. (2014). Joint temporal and contemporaneous aggregation of random-coefficient AR(1) processes. *Stochastic Process. Appl.* **124**, 1011–1035.

[6] Robinson, P.M. (1978). Statistical inference for a random coefficient autoregressive. *Scand. J. Stat.* **5**, 163–168.

THANK YOU FOR YOUR ATTENTION