Lévy based modelling in stochastic geometry and spatial statistics

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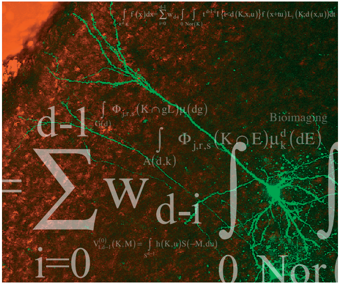

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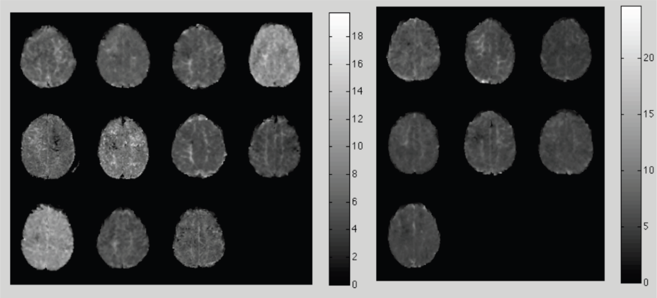
Lévy based modelling

Lévy based modelling is a flexible, yet tractable modelling tool.

- Lévy based random fields - tractable extension of Gaussian random fields (neuroscience applications)
- Lévy particles - tractable model for spatial particles of varying shape (including growth)
- Lévy based Cox point processes - unification of existing models and creation of some new ones

Example from brain imaging



MTT brain scan images with data $X_{i,j,t}$:

- $i = 1, 2$ (groups of subjects)
- $j = 1, \ldots, n_i$ (subjects within groups)
- $t \in V$ (voxels of interest)

Standard method for group comparison

- A measure for the group difference is

$$T_t = \frac{\bar{X}_{1 \cdot t} - \bar{X}_{2 \cdot t}}{\sqrt{S_t^2 \left(\frac{1}{n_1} + \frac{1}{n_2} \right)}},$$

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- Consider the field of test statistics

$$T = \{T_t \mid t \in V\}.$$

Many voxels will be falsely declared significant if a marginal threshold at each voxel is applied.

Using the maximum of the field for inference

- Find the probability that the maximum of the random field T exceeds a certain value

$$\mathbb{P}(\max_{t \in V} T_t > x_\alpha) = \alpha \qquad (\text{say } \alpha = 0.05)$$

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  - The original data needs to be Gaussian or at least the T–image should be t–distributed
  - That is not necessarily the case...

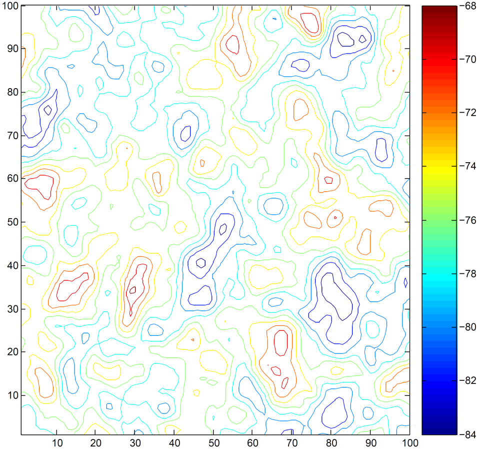
Lévy based random fields

- We consider a Lévy based random field

$$X_t = \int_{\mathbb{R}^2} k(t,s)\, M(\mathrm{d}s), \quad t \in \mathbb{R}^2,$$

  where M is a Lévy basis and k is a kernel function.
- M can e.g. be Gaussian, Gamma, inverse Gaussian, normal inverse Gaussian (NIG), ...
- The cumulant function of X_t can be calculated using the cumulant function for M.
- It is easy to calculate the cumulants/moments of X_t.

Simulated NIG random field



Correlation structure

We assume that k has the form

$$k(t,s) = k(||t-s||)\ .$$

k determines the correlations $\mathrm{Corr}(X_{t_1}, X_{t_2}) = \rho(||t_1 - t_2||)$:

- Exponential correlation model

$$k(t,s) = \frac{\sigma^2}{4\pi||t-s||}\mathrm{e}^{-\sigma||t-s||} \quad \Rightarrow \quad \rho(d) = \mathrm{e}^{\sigma d}$$

- Matérn correlation model

$$k(t,s) = K||\alpha(t-s)||^{\nu/2-3/4}K_{\nu/2-3/4}(\alpha||t-s||)$$
$$\Rightarrow \quad \rho(d) = C(\alpha d)^{\nu}K_{\nu}(\alpha d)$$

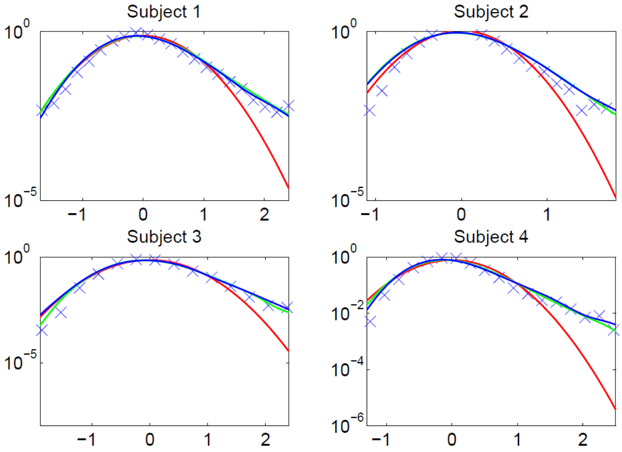


Figure: Log–histograms for data (×) together with fitted NIG field (blue and green) and Gaussian densities (red). Jónsdóttir *et al.* (2013, *Scand. J. Stat.*

Recall the model assumption

$$X_t = \int_{\mathbb{R}^2} k(\|t-s\|)\, M(\mathrm{d}s), \quad t \in \mathbb{R}^2.$$

The Lévy basis M satisfies the Lévy-Khintchine representation:

$$\log \mathbb{E}[\mathrm{e}^{i\lambda M(A)}] = i\lambda a|A| + |A| \int_{\mathbb{R}} \left(\mathrm{e}^{i\lambda u} - 1 - i\lambda u 1_{[-1,1]}(u)\right) \nu(\mathrm{d}u)$$

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We furthermore assume $\nu((x,\infty)) \sim Cx^{-\delta}e^{-\beta x}$ as $x \to \infty$ with $\delta > 1$.

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We are interested in the tail behaviour of $P(\sup_{t\in B} X_t > x)$ as $x \to \infty$, where B is a bounded closed subset of $\mathbb{R}^2$.

Theorem

For a computable constant K, we have

$$P(\sup_{t \in B} X_t > x) \sim K \cdot E\left(e^{\beta X_{t_0}}\right) x^{-\delta} \exp(-\beta x),$$

as $x \to \infty$, with $t_0 \in B$ arbitrarily chosen.

Rønn-Nielsen & J. (2014)

Under publication in *Adv. Appl. Prob.*

Joint work with Johanna F. Ziegel.

- We consider a Lévy based stochastic process on the unit sphere $\mathbb{S}^2$

$$X_u = \int_{\mathbb{S}^2} k(u,v)\, M(\mathrm{d}v), \quad u \in \mathbb{S}^2,$$

  where M is a Lévy basis on $\mathbb{S}^2$ and k is a kernel function.
- This process is used in the defininition of Lévy particles which are random deformations of a fixed particle $K_0 \subset \mathbb{R}^3$.

The Lévy particles $K \subset \mathbb{R}^3$ are star-shaped with respect to a fixed point in $\mathbb{R}^3$, taken to be the origin O.

Let R_u, $u \in \mathbb{S}^2$, be the radial function of K, i.e. R_u is the distance from O to the boundary of K in direction u.

Then,

$$R_u = c_u X_u = c_u \int_{\mathbb{S}^2} k(u,v)\, M(\mathrm{d}v), \quad u \in \mathbb{S}^2,$$

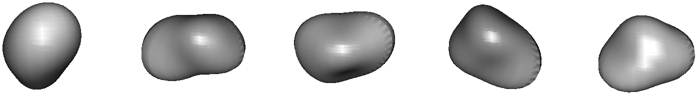
where $\{c_u : u \in \mathbb{S}^2\}$ is the radial function of the fixed particle K_0.

Lévy particles

Simulated Lévy particles obtained as random deformations of an ellipsoid K_0, using a von Mises-Fisher kernel

$$k(u, v) = e^{\alpha\, d(u,v)}, \quad u, v \in \mathbb{S}^2,$$

and a Gamma Lévy basis M.



Parameter estimation

A method of moments for the so-called particle cover density may be used.

The particle cover density is the probability density on $\mathbb{R}^3$ given by

$$f_K(x) = \mathbb{P}(x \in K)/\mathbb{E}V(K), \quad x \in \mathbb{R}^3,$$

where $V(K)$ is the volume of K.

The density f_K indicates how likely it is for a point in space to be hit by the Lévy particle K.

Moment relations

Recall that the radial function of K is of the form

$$R_u = c_u X_u = c_u \int_{\mathbb{S}^2} k(u,v)\, M(\mathrm{d}v), \quad u \in \mathbb{S}^2,$$

where $\{c_u : u \in \mathbb{S}^2\}$ is the radial function of the fixed particle K_0.

Let $\mu_k = \mathbb{E}(X_u^k)$. Then,

$$\mathbb{E}V(K) = \mu_3 V(K_0).$$

The parameters of the process X_u is chosen such that $\mu_3 = 1$.

Moment relations

The mean μ of the cover density becomes O, due to the choice of the reference point of K.

The covariance matrix Σ of the cover density becomes

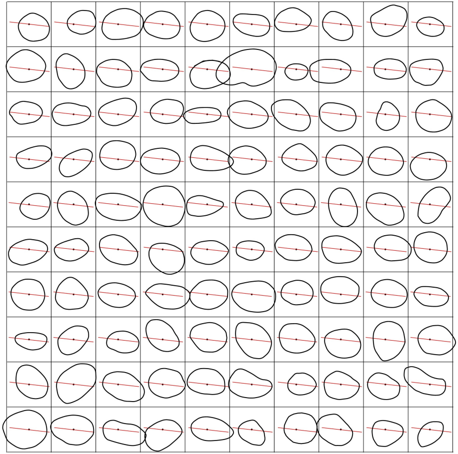
$$\Sigma = \frac{\mu_5}{\mu_3} \frac{1}{V(K_0)} \int_{K_0} x^2 \, \mathrm{d}x,$$

where x^2 is the symmetric tensor of rank 2 induced by x.

Using the two red equations, the parameters of K_0 and the process X_u can be estimated.

Lévy particles

Profiles from 100 sampled neuron nuclei from brain cortex



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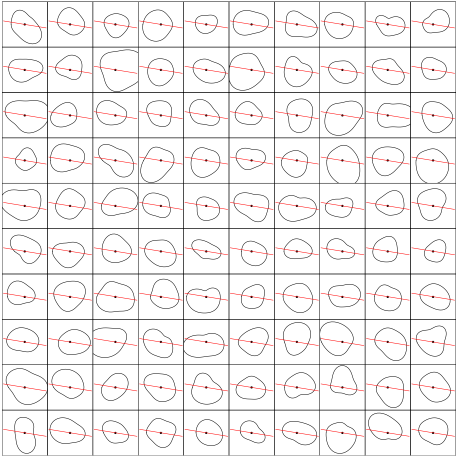
Lévy particles

Profiles from 100 simulated particles under the fitted model



Lévy particles are useful in simulation studies

Recently, non-parametric methods for estimating shape and orientation of arbitrary particles have been developed in Ziegel, Nyengaard & J. (2015, *Scand. J. Stat.*).

An efficient way of studying the statistical behaviour of these methods is to apply them on a range of simulated Lévy particles.

Lévy based growth models

It is easy to extend the model associated with Lévy particles to a spatio-temporal model

$$X_{u,t} = \int_{A_t(u)} k(u,t;v,s)M(\mathrm{d}(v,s)), \quad u \in \mathbb{S}^2, t \in \mathbb{R},$$

where $A_t(u) \subseteq \mathbb{S}^2 \times (-\infty, t]$ is an ambit set associated with each $(u,t) \in \mathbb{S}^2 \times \mathbb{R}$ and M is a Lévy basis on $\mathbb{S}^2 \times \mathbb{R}$.

Growth relates to $\frac{\partial}{\partial t} X_{u,t}$.

Jónsdóttir, Schmiegel & J. (2008, *Bernoulli*).

Joint work with Michaela Prokešová and Gunnar Hellmund.

Cox point processes - short reminder

Notation:

Φ, spatial point process on $\mathbb{R}^d$.

$\{X_u : u \in \mathbb{R}^d\}$, random field on $\mathbb{R}^d$.

Definition Let $\{X_u : u \in \mathbb{R}^d\}$ be a non-negative almost surely locally integrable random field. A point process Φ on $\mathbb{R}^d$ is a Cox point process with the driving field X if conditionally on X, Φ is a Poisson process with intensity function X.

Definition The Lévy based Cox point process (LCP) has driving field of the form

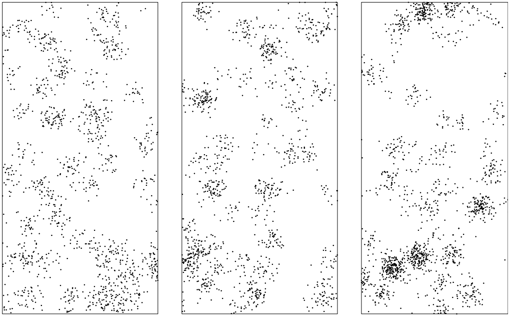
$$X_u = \int_{\mathbb{R}^d} k(u,v)M(\mathrm{d}v), \qquad u \in \mathbb{R}^d, \tag{1}$$

where M is a non-negative Lévy basis and k is a non-negative function such that $k(u,\cdot)$ is integrable w.r.t. M for each u and $k(\cdot,v)$ is integrable w.r.t. the Lebesgue measure on $\mathbb{R}^d$, for each v.

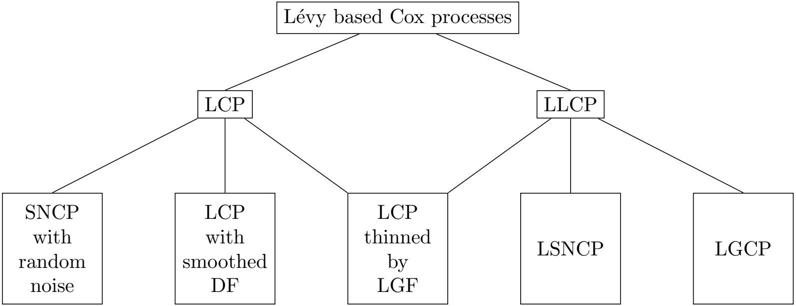
For the LCP to be well defined k and M are chosen in such a way that X is locally integrable almost surely.

The influence of the spot variable

Stationary LCPs with spot variable being (from left to right) Poisson, gamma and inverse Gaussian distributed.



Overview


Lévy based Cox processes
LCP
LLCP
SNCP with random noise
LCP with smoothed DF
LCP thinned by LGF
LSNCP
LGCP