Sand, Wind and Stochastics

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The Sand Gang in the 1970s

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Ralph Alger Bagnold, 1896 – 1990

The hyperbolic distribution - 1941

Bagnold (1941): The Physics of Blown Sand and Desert Dunes, p. 115

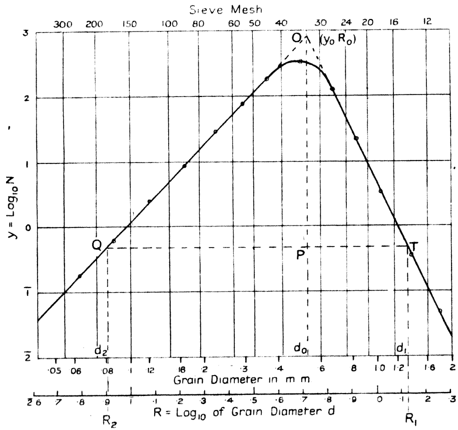

FIG. 33.—THE LOG DIAGRAM

The hyperbolic distribution - 1977

Barndorff-Nielsen (1977):

$$\frac{\gamma}{2\alpha\delta K_1(\delta\gamma)} \exp\left\{-\alpha\sqrt{\delta^2+(x-\mu)^2}+\beta(x-\mu)\right\}$$

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The generalized hyperbolic distribution:

$$\frac{(\gamma/\delta)^\lambda}{\sqrt{2\pi}K_\lambda(\delta\gamma)} \cdot \frac{K_{\lambda-\frac{1}{2}}\left(\alpha\sqrt{\delta^2+(x-\mu)^2}\right)}{\left(\sqrt{\delta^2+(x-\mu)^2}/\alpha\right)^{\frac{1}{2}-\lambda}} \cdot e^{\beta(x-\mu)}$$

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The normal-inverse Gaussian (NIG) distribution:

$$\frac{\alpha\delta}{\pi}e^{\delta\gamma} \cdot \frac{K_1\left(\alpha\sqrt{\delta^2+(x-\mu)^2}\right)}{\sqrt{\delta^2+(x-\mu)^2}} \cdot e^{\beta(x-\mu)}$$

Size distributions: a breakage model, Kolmogorov(1947)

Grain size after N breakage events:

$$S_N = s_0 \prod_{i=1}^{N} (1 - D_i)$$

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$$\log S_N = \mu + \sum_{i=1}^{N} B_i, \qquad \text{where } \mu = \log(s_0),\ B_i = \log(1 - D_i)$$

$$\frac{1}{\sqrt{N}} \left(\sum_{i=1}^{N} B_i - N b_1 \right) \xrightarrow{\mathcal{D}} N(0, b_2)$$

as $N \to \infty$, where $b_1 = E(B_i)$ and $b_2 = E(B_i^2)$

$$\log S_N \mathrel{\dot\sim} N(\mu + N b_1, N b_2)$$

Size distributions: a breakage model, 2015

Grain size at time t: $$S_t = s_0 \prod_{i=1}^{N_t} (1 - D_i)$$

N_t number of events that cause breakage, Poisson (λ)

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$$U_\lambda(t) = \frac{1}{\sqrt{\lambda}} \left(\sum_{i=1}^{N_t} B_i - \lambda t b_1 \right) \xrightarrow{\mathcal{D}} N(0, tb_2)$$

as $\lambda \to \infty$, where $b_1 = E(B_i)$ and $b_2 = E(B_i^2)$

$$\log S_t \;\dot{\sim}\; N(\mu + \lambda t b_1, \lambda t b_2)$$

Size distributions: a breakage model, 2015

Suppose grains move according to a Brownian motion with drift ν and diffusion coefficient σ^2

Then a grain arrives at the deposit at time $\tau \sim IG(a/\sigma, \nu/\sigma)$,

$a =$ distance between the source and the deposit

Size distributions: a breakage model, 2015

Suppose grains move according to a Brownian motion with drift ν and diffusion coefficient σ^2

Then a grain arrives at the deposit at time $\tau \sim IG(a/\sigma, \nu/\sigma)$,

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$$\log S_\tau \,|\, \tau = t \,\dot{\sim}\, N(\mu + \lambda b_1 t, \lambda b_2 t)$$

when λ is sufficiently large, so

$$\log S_\tau \,\dot{\sim}\!= \mathrm{NIG}(\alpha, \beta, \delta, \mu),$$

where

$$\begin{aligned} \alpha &= \sqrt{b_1^2/b_2^2 + \nu^2/(\sigma^2 \lambda b_2)} \\ \beta &= b_1/b_2 \\ \delta &= \frac{a}{\sigma}\sqrt{\lambda b_2} \end{aligned}$$

Size distributions: a breakage model, 2015

$\bar{t} = a/\nu$ is the average travel time of a grain

Define $\theta = \lambda \bar{t} b_1 \qquad \omega = \sqrt{\bar{t}\sigma^2}/(\bar{t}\nu) \qquad c_0 = \sqrt{b_2}/b_1$

$$\begin{aligned}
\text{mean} &= \mu + \theta \\
\text{variance} &= \theta b_1 (\lambda \bar{t} \omega^2 + c_0^2) \\
\chi &= -\frac{1}{\sqrt{(1 + c_0^2/(\lambda \bar{t} \omega^2))\,(1 + \omega^{-2})}} \\
\xi &= \frac{1}{\sqrt{1 + \omega^{-2}}}
\end{aligned}$$

The transport rate and air borne shear stress

TRANSPORT RATE: $Q = \Phi \cdot$ MEAN JUMP LENGTH

Φ = MASS FLUX FROM THE BED INTO THE AIR

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Owen (1964)

$$\begin{aligned} \rho U_*^2 &= \text{AIR BORNE SHEAR STRESS + GRAIN BORNE SHEAR STRESS} \\ &= T_a(y) + T_g(y) \end{aligned}$$

U_* shear velocity in the grain free wind, ρ density of air

Grain borne shear stress at height y: $T_g(y) = \Phi v(y)$

$v(y)$ = the average increase of the horizontal velocity component of a saltating grain while it is above the level y

A simple saltation model: the wind

AIR BORNE SHEAR STRESS AT THE BED: $T_a(0) = \pi\rho U_*^2, \quad 0 < \pi < 1$

A simple saltation model: the wind

AIR BORNE SHEAR STRESS AT THE BED: $T_a(0) = \pi \rho U_*^2, \quad 0 < \pi < 1$

It follows that

$$\Phi = \rho U_*^2 \left(1 - \pi\right) / v(0)$$

$$T_g(y) = \rho U_*^2 \left(1 - \pi\right) a(y), \qquad a(y) = v(y)/v(0)$$

By eddy viscosity/Prandtl turbulence closure (and an approximation):

$$U(y) = \kappa^{-1} U_* \left[\ln(y/y_0) - \left(1 - \sqrt{\pi}\right) b(y) \right]$$

where $\kappa =$ von Kármán's constant and

$$b(y) = \int_{y_0}^{y} z^{-1} a(z) dz$$

A simple saltation model: the grain motion

$$\ddot{x} = H(v)(U(y) - \dot{x})$$
$$\ddot{y} + g + H(v)\dot{y} = 0$$

$(x(0), y(0)) = (0, 0)$ $\quad$ $(\dot{x}(0), \dot{y}(0)) = (v_1^0, v_2^0)$ is random

$H(v) = D(v)/(mv)$

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Owen (1964): $\quad D(v) = \delta v \quad H(v) = t_*^{-1}$

t_* is the response time of a grain to changes in the wind speed

$$x(t) = \int_0^t \left(1 - e^{-(t-s)/t_*}\right) U(y(s))ds + t_* v_1^0 \left(1 - e^{-t/t_*}\right)$$
$$y(t) = t_*(v_f + v_2^0)(1 - e^{-t/t_*}) - v_f t \qquad v_f = gt_*$$

A simple saltation model: the transport rate

$Q = \Phi \cdot \overline{x(t_i)}$ t_i impact time

$$\overline{x(t_i)} = \kappa^{-1} U_* \left[\overline{\int_0^{t_i} (1 - e^{-(t_i - s)/t_*}) \log(y(s)/y_0) ds} - (1 - \sqrt{\pi}) \overline{\int_0^{t_i} (1 - e^{-(t_i - s)/t_*}) b(y(s)) ds} \right] + t_* \overline{v_1^0 (1 - e^{-t_i/t_*})}.$$

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$$\Phi = \rho U_*^2 \left(1 - \pi\right) / v(0)$$

$$\frac{Qg}{\rho U_*^3} = (1 - \pi) \left[\alpha + \beta \sqrt{\pi} + \gamma / U_*\right]$$

Bagnold (1941): $Q \propto \rho g^{-1} U_*^3$

A simple saltation model: the transport rate

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Sørensen (1991, 2004, 2013), Durán and Herrmann (2006)

Owen (1964)

AIR BORNE SHEAR STRESS AT THE BED IS EQUAL TO ρU_{*c}^2 FOR ALL $U_* \geq U_{*c}$

U_{*c} shear velocity at the impact threshold

$$\pi = \frac{U_{*c}^2}{U_*^2} = V^{-2}$$

$$V = \frac{U_*}{U_{*c}} \qquad \text{dimensionless shear velocity}$$

$$\frac{Qg}{\rho U_*^3} = \left(1 - V^{-2}\right)\left[\alpha + \beta V^{-1}\right]$$

Keld Rømer Rasmussen

A simple saltation model: lift-off velocity

$(\dot{x}(0), \dot{y}(0)) = (v_1^0, v_2^0)$ is random

Probability density of v_2^0:

$$\rho \lambda_s e^{-\lambda_s x} + (1-\rho)\lambda_r e^{-\lambda_r x}, \qquad 0 < \rho < 1$$

Saltators: $E(v_1^0 \mid v_2^0) = \kappa v_2^0 \qquad \kappa = \cot(\alpha), \quad \alpha = 30$

Reptators: $\overline{v_1^0} = 0$

Model fitted by least squares to data for grains of size 310 μ:

Horizontal and vertical velocity components of grains going up and grains going down at 9 heights: 0.3, 0.5, 0.8, 1.5, 2.0, 2.5, 3.0, 4.5, 6 cm

4 shear velocities: 31, 44, 66 and 103 cm/s

A simple saltation model: parameter estimates

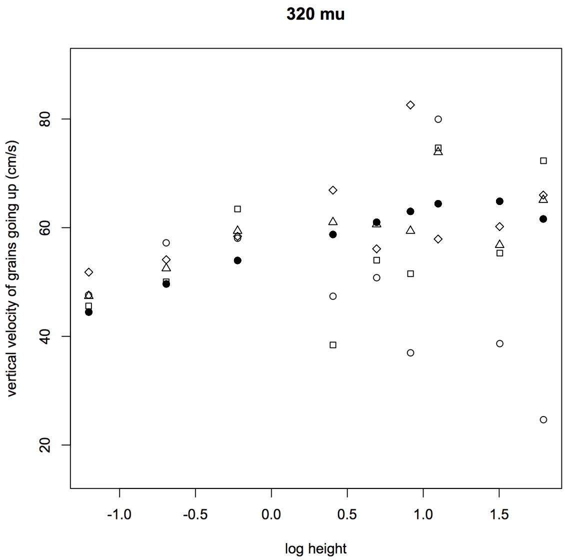
$\rho = 0.33$

Mean number of grains ejected by an impinging saltator: $\bar{n} = \frac{1-\rho}{\rho} = 2.02$

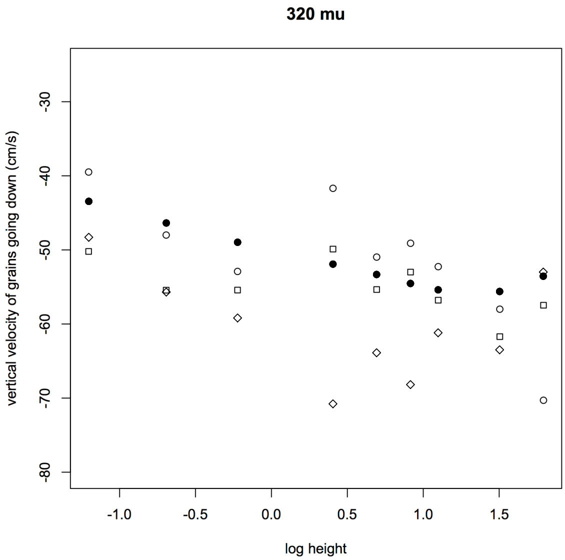
$1/\lambda_s = 31$ cm/s

$1/\lambda_r = 7$ cm/s

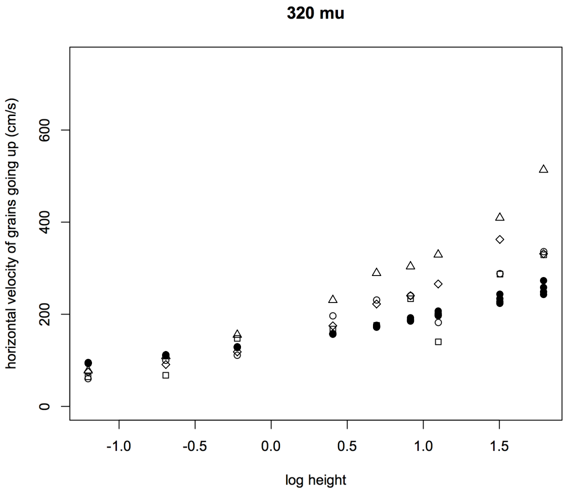
Vertical velocity component of grains going up



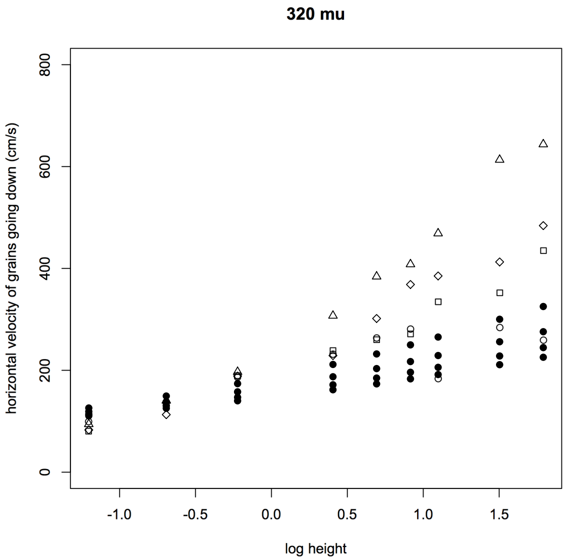
Vertical velocity component of grains going down



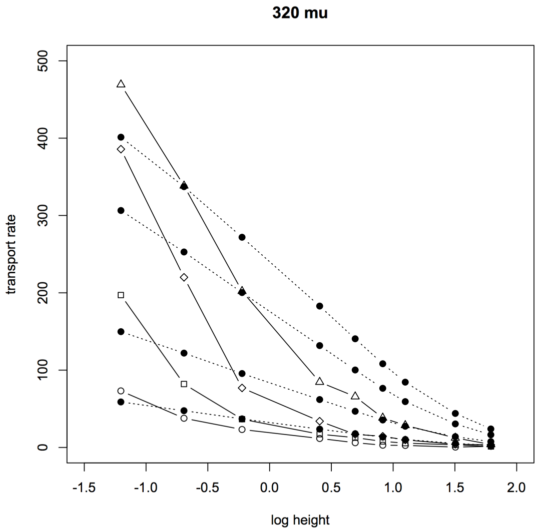
Horizontal velocity component of grains going up



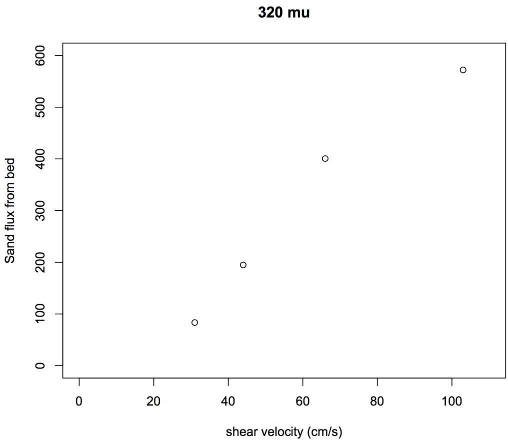
Horizontal velocity component of grains going down



Transport rate profile

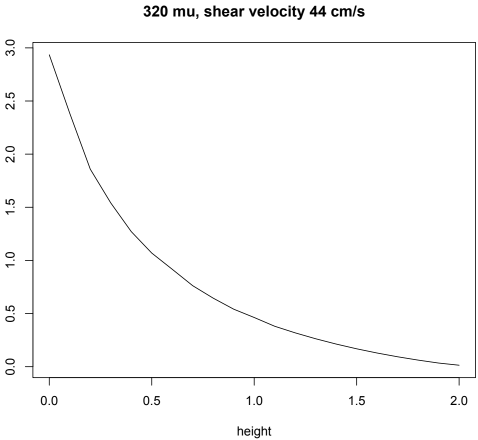


Flux from bed to air



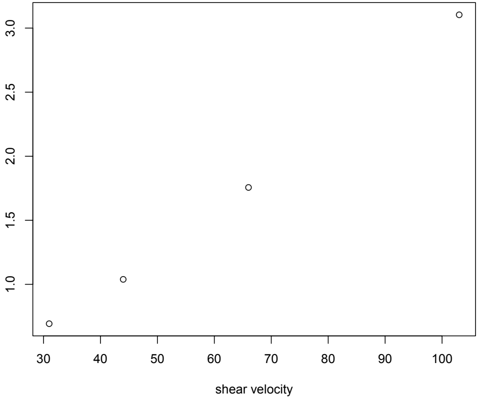
$$\Phi = \rho U_*^2 \,(1 - \pi)\,/\,v(0)$$

Grain borne shear stress profile



$$T_g(y) = \Phi v(y)$$

Shear velocity at the bed



Owen (1964) hypothesised that the air borne shear stress at the bed is constantly equal to ρU_{*c}^2 for all $U_* \geq U_{*c}$

THANK YOU OLE!

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CONGRATULATIONS!