Exponential functionals of Lévy and Markov additive processes

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[1] Joint work with Jonas Arista (first part); and Andreas Kyprianou and Weerapat Satitkanitkul (second part)

Let ξ be a $\mathbb{R}$-valued Lévy process killed at an independent exponential time of parameter $q \geq 0$. $-\infty$ is the cemetery state and

$$\zeta = \inf\{t > 0 : \xi_t = -\infty\}.$$

The law of ξ is characterised by

$$\mathbb{E}\left(\exp\{i\lambda\xi_t\}, t < \zeta\right) = \exp\{-t\Psi(\lambda)\}, \qquad t \geq 0, \lambda \in \mathbb{R},$$

where $\Psi : \mathbb{R} \to \mathbb{C}$, is the characteristic exponent and has the form

$$\Psi(\lambda) = q + ia\lambda + \frac{1}{2}\sigma^2\lambda^2 + \int_{\mathbb{R}\setminus\{0\}} \left(1 - e^{i\lambda x} + i\lambda x\right) \Pi(dx), \qquad \lambda \in \mathbb{R},$$

with $q \geq 0$, $a \in \mathbb{R}$, $\sigma \in \mathbb{R}$, and Π is a measure on $\mathbb{R}\setminus\{0\}$ such that

$$\int_{\mathbb{R}\setminus\{0\}} 1 \wedge x^2 \Pi(dx) < \infty.$$

(q, a, σ^2, Π) is the *characteristic quadruple*.

We assume that either

$$q > 0$$

or

$$q = 0 \qquad \text{and} \quad \lim_{t\to\infty} \xi_t = -\infty, \qquad \mathbf{P}\text{ -a.s.}$$

Under these conditions the *exponential functional* associated to ξ, is finite a.s.

$$I := \int_0^\infty e^{\xi_s} ds < \infty, \qquad \mathbf{P}\text{ -a.s.}$$

The strong law of large numbers implies that, if $q = 0$, ξ grows at least linearly, and hence the above are NASC for I to be finite a.s.

Applications to:

- Self-similar Markov processes;
- Brownian diffusions in random environment;
- Mathematical finance, computation of price of asian options;
- Self-similar fragmentations and coalescence;
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Ignited by a seminal paper by Carmona, Petit and Yor in 1997 the obtainment of distributional properties of exponential functionals has generated a big research activity in the last two decades.

We will start by reviewing some general results available for its study. Then we will introduce a new convolution equation for the density of I and explain several consequences.

An integro-differential equation

Carmona, Petit and Yor, 1997, established that if ξ has an infinite lifetime, its jump part has bounded variation,

$$\int_{x>1} x\overline{\Pi}^{+}(x)dx < \infty, \quad \text{with } \overline{\Pi}^{+}(x) = \Pi(x,\infty),\ \overline{\Pi}^{-}(x) = \Pi(-\infty,-x),$$

for $x > 0$, and $\mathbf{E}(|\xi_1|) < \infty$, then the law of I has a density, k, which is the unique probability density function that solves the equation

$$\begin{aligned} &-\frac{\sigma^2}{2}\frac{d}{dx}\left(x^2 k(x)\right) + \left(\left(\frac{\sigma^2}{2} + a\right)x + 1\right)k(x) \\ &\quad = \int_x^\infty \overline{\Pi}^{-}(\log(u/x))k(u)du - \int_0^x \overline{\Pi}^{+}(\log(x/u))k(u)du, \qquad x > 0. \end{aligned}$$

Extensions of this equation to general Lévy processes and to exponential functionals of the form $\int_0^\infty \exp\{\xi_s\}d\eta_s$, with (ξ,η) a Lévy processes, have been obtained in papers by Behme, Lindner, Kuznetsov, Pardo, Patie, Savov...

In general, it is difficult to extract from this integro-differential equation an explicit formula for the density. Nevertheless it has been useful in determining the behaviour at zero of the distribution of I, as well as that of its tail distribution, in the case where ξ is the negative of a subordinator, see Pardo, R. and Van Schaik 2012 and Haas and R. 2012, respectively.

Moments

- Carmona, Petit and Yor in 1997, Maulik and Zwart in 2006,...., established a recurrence formula for the moments of I:

$$\mathbf{E}(I^{\beta}) = \frac{\beta}{\Psi(-i\beta)}\,\mathbf{E}(I^{\beta-1}),$$

for $\beta \in \mathbb{C}$ such that $\Re(\beta) > 0$ and $|\,\mathbb{E}(e^{\beta\xi_1}1_{\{1<\zeta\}})| < 1$.

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- This formula has proven very useful to write the Mellin transform of I as generalised Weierstrass pruducts, see Maulik and Zwart (2006) and Patie and Savov (2013), and also to obtain series expansions for the density of I for special families of Lévy processes, Patie (2009) and Kuznetsov and Pardo (2013).

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- It also allows to infer interesting factorisations that we will next describe.

The moment formula in the non-increasing case

- If $\xi = -\sigma$, with σ a subordinator (non-decreasing paths), $\phi(\lambda) = -\log \mathbb{E}(\exp\{-\lambda\sigma_1\}1_{\{1<\zeta\}})$,

$$\mathbb{E}(I^n) = \prod_{k=1}^{n} \frac{k}{\phi(k)} = \frac{n!}{\prod_{k=1}^{n} \phi(k)}, \qquad n \geq 0,$$

and $\mathbb{E}(\exp\{\beta I\}) < \infty$, for $\beta < \phi(\infty)$.

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- The law of I is characterised by its entire moments.
- Since $n! = \mathbb{E}\left(\mathbf{e}^n\right)$, with $\mathbf{e}$ is an exponential r.v. one may wonder if there is a r.v., say R such that $\mathbb{E}(R^n) = \prod_{k=1}^{n}\phi(k)$, for all n. In the positive case, if I and R are independent then

$$I \times R \stackrel{\text{Law}}{=} \mathbf{e}.$$

The residual exponential functional

Proposition (Bertoin and Yor (2001))

Assume that $\xi = -\sigma$, *with* σ *a subordinator and take* $I = \int_0^\zeta e^{-\sigma_s} ds$. *There exists a r.v.* R *determined by its entire moments and*

$$\mathbb{E}(R^\lambda) = \phi(\lambda)\, \mathbb{E}(R^{\lambda-1}), \qquad \lambda > 0.$$

The identity holds in the limit sense if $\lambda = 0$. *In particular,*

$$\mathbb{E}(R^n) = \prod_{k=1}^{n} \phi(k), \qquad n \in \mathbb{N}\,.$$

If R *and* I *are taken independent then*

$$I R \stackrel{\text{Law}}{=} \mathbf{e}_1 \sim \exp\{1\}.$$

The random variable R *is called the* **residual exponential functional** *associated to the subordinator* σ.

Theorem (Hirsch and Yor (2011), Arista & R. (2015))

Assume $\xi = -\sigma$, and σ is a subordinator. Let R be the residual exponential functional associated to the subordinator σ, and

$$V(dy) = \mathbb{E}\left(\int_0^{\zeta} dt 1_{\{\sigma_t \in dy\}}\right), \qquad y \geq 0.$$

Assume $\phi(0) > 0$, that is $\mathbb{P}(t < \zeta) = \exp\{-\phi(0)t\}, t \geq 0$. The equality of measures

$$\frac{1}{t}\,\mathbb{P}(R \in dt) = \int_{[0,\infty)} V(dy)\,\mathbb{P}(e^{-y}R \in dt), \qquad \text{on } (0,\infty),$$

holds. In particular $\mathbb{E}\left(\frac{1}{R}\right) = \phi(0)$ and the random variable J_σ defined by

$$\mathbb{E}\left(f(J_\sigma)\right) = \phi(0)\,\mathbb{E}\left(f\left(\frac{1}{R}\right)\frac{1}{R}\right),$$

satisfies the relation $J_\sigma \stackrel{Law}{=} e^{\nu} \times \frac{1}{R}$, with $\mathbb{P}(\nu \in dy) = \phi(0)V(dy)$.

A striking factorisation

Pardo, Patie and Savov in three papers in 2012-2013 used the recurrence of moments to show that

$$I \stackrel{\text{Law}}{=} \int_0^\infty \exp\{-\widehat{H}_s\}ds \times J_H,$$

where the processes $(\widehat{H}_s, s \geq 0)$ and $(H_s, s \geq 0)$ are assumed independent and are copies of the so-called downward and upward ladder height process of ξ, respectively. $-\widehat{H}$ has the same image as the infimum process of ξ. H has the same image as the supremum process of ξ. Where R_H is the unique r.v. such that

$$R_H \times \int_0^\infty \exp\{-H_s\}ds \stackrel{\text{Law}}{=} \mathbf{e}_1,$$

and

$$\mathbb{E}\left(f(J_H)\right) = \phi_H(0)\,\mathbb{E}\left(f\left(\frac{1}{R_H}\right)\frac{1}{R_H}\right),$$

The Wiener-Hopf factorisation implies that the characteristic exponent of ξ, say Ψ, can be written

$$\frac{1}{\Psi(\lambda)} = \frac{1}{\kappa(q,-i\lambda)}\frac{1}{\widehat{\kappa}(q,i\lambda)}, \qquad \lambda \in \mathbb{R},$$

with $\kappa(q,\cdot)$ and $\widehat{\kappa}(q,\cdot)$ are the Laplace exponent of H and $\widehat{H}$. This plugged into the moment formula

$$\mathbf{E}(I^{\beta}) = \frac{\beta}{\Psi(-i\beta)}\mathbf{E}(I^{\beta-1}) = \frac{1}{\kappa(q,-\beta)}\frac{\beta}{\widehat{\kappa}(q,\beta)}\mathbf{E}(I^{\beta-1}),$$

$$\mathbf{E}\left(I^{\beta}_{\widehat{H}}\right) = \frac{\beta}{\widehat{\kappa}(q,\beta)}\mathbf{E}\left(I^{\beta-1}_{\widehat{H}}\right),$$

with $I_{\widehat{H}} = \int_0^{\zeta}\exp\{-\widehat{H}_s\}ds$, allowed Pardo, Patie and Savov (2012) to infer that $I_{\widehat{H}}$ should be involved in a factorisation of I. The identification of the factor J_H was noticed in the paper by Patie and Savov (2013). The proof in the general case is based in an expression of the Mellin transform of I in infinite products.

Aims

- Have a better understanding of how the Wiener-Hopf factorisation gives rise to the factorisation

$$I \overset{\text{Law}}{=} \int_0^\infty \exp\{-\widehat{H}_s\} ds \times J_H.$$

- Establish other proofs that could lead to extensions to processes with a similar estructure to Lévy processes, namely Markov additive process.

A measure version of the CPY moment formula

Theorem

Let $U(dy)$ be the renewal or potential measure of ξ,

$$U(dy) = \mathbf{E}\left(\int_0^\zeta dt 1_{\{\xi_t \in dy\}}\right), \qquad y \in \mathbb{R}.$$

The law of I has a density k, and it is the unique probability density function on $(0,\infty)$ that solves the equation

$$\int_t^\infty k(s)ds = \int_{\mathbb{R}} k(te^{-y})U(dy), \qquad \text{on } (0,\infty).$$

In the particular case where ξ is the negative of a subordinator this theorem has been established by Pardo, R. and Van Schaik (2012). The proof in the general case is rather simple. This existence of the density was proved by Bertoin, Lindner and Maller in 2008.

The proof uses three facts

- $\mathbb{E}(1-e^{-\lambda I}) = \lambda \int_0^\infty e^{-\lambda t}\, \mathbb{P}(I > t) dt.$
- the pathwise identity

$$1 - \exp\{-\lambda \int_0^\zeta e^{\xi_s} ds\} = \lambda \int_0^\zeta dt e^{\xi_t} \exp\left\{-\lambda \int_t^\zeta e^{\xi_s} ds\right\}$$
$$= \lambda \int_0^\zeta dt e^{\xi_t} \exp\left\{-\lambda e^{\xi_t} \int_0^{(\zeta - t)} e^{\xi_{s+s} - \xi_t} ds\right\}.$$

- the property of independent and stationary increments implies that the r.v.

$$\widetilde{I} := \int_0^{(\zeta - t)} e^{\xi_{s+s} - \xi_t} ds,$$

on the event where $t < \zeta$, has the same law as I and it is independent of $(\xi_u, u \leq t)$.

Applications

The equation

$$\int_t^\infty k(s)ds = \int_{\mathbb{R}} k(te^{-y})U(dy), \qquad \text{on } (0,\infty).$$

Can be applied to:

- give a quick proof of the Carmona et al. recurrence formula for the moment of I (just take the Mellin transform);
- derive a general version of the integro-differential formula of Carmona et al.
- give an elementary proof of the factorisation formula of Pardo, Patie and Savov;
- obtain asymptotic estimates for the distribution and tail distribution of I using renewal theoretic arguments.

Moreover, the same proof allows to extend this formula for Markov additive processes (MAPs) and traces the path for extending the above results for MAPs.

Let H, and $\widehat{H}$, be the upward, resp. downward, ladder height processes of ξ.

- The so Wiener-Hopf factorisation in space implies that U is the negative convolution of the potentials of H, and $\widehat{H}$,

$$V^{+}(dy) = \mathbf{E}\left(\int_0^\infty dt 1_{\{H_t \in dy\}}\right), \; V^{-}(dy) = \mathbf{E}\left(\int_0^\infty dt 1_{\{\widehat{H}_t \in dy\}}\right),$$

$$\int_{\mathbb{R}} U(dy) f(y) = K \int_{[0,\infty)} \int_{[0,\infty)} V^{+}(du) V^{-}(dv) f(u-v),$$

for every $f : \mathbb{R} \to \mathbb{R}$ test function, and some constant $0 < K < \infty$.

- In our setting H has finite lifetime, the measure V^{+} is a finite measure and the overall supremum of ξ, $S_\infty = \sup_{0 \leq s < \zeta} \xi_s$, has law $\mathbb{P}(S_\infty \in dy) = \kappa V^{+}(dy), \qquad y \geq 0.$
- If ξ has no negative jumps, $\Pi(-\infty, 0) = 0$, $V^{-}(dv) = dv$.
- If ξ has no positive jumps $\Pi(0, \infty) = 0$, $V^{+}(du) = e^{-\theta u} du$, with θ s.t. $\mathbb{E}(e^{\theta \xi_1}, 1 < \zeta) = 1$.

A proof of the identity by Pardo, Patie and Savov

- We know $I = \int_0^\infty \exp\{\xi_s\} ds$
  $$\mathbb{P}(I > t) = \int_{[0,\infty)} V^+(da) \int_{[0,\infty)} V^-(db) k_I(te^{b-a}), \qquad t > 0.$$
- Let $\widehat{I}_{\widehat{H}} = \int_0^\infty \exp\{-\widehat{H}_s\} ds$, the equation for this r.v. reads
  $$\mathbb{P}(I_{\widehat{H}} > t) = \int_{[0,\infty)} V^-(du) k_{I_{\widehat{H}}}(te^u).$$
- Let $\widehat{I}_H = \int_0^\infty \exp\{-H_s\} ds$, and R_H the associated residual exponential functional, i.e. the unique r.v. such that $\widehat{I}_H R_H \stackrel{\text{Law}}{=} \mathbf{e}_1$.
  We saw that $\frac{1}{t}\mathbb{P}(R_H \in dt) = \int_{[0,\infty)} V^+(dy)\,\mathbb{P}(e^{-y} R_H \in dt)$ and $\mathbb{P}(\sup_{t\geq 0} \xi_t \in dy) = \kappa V^+(dy)$.
- Using these facts, the identity in law of Pardo, Patie and Savov
  $$I \stackrel{\text{Law}}{=} \widehat{I}_{\widehat{H}} J_H \stackrel{\text{Law}}{=} e^{S_\infty} \frac{\widehat{I}_{\widehat{H}}}{R_H},$$
  is obtained easily. $\mathbb{E}\left(f(J_H)\right) = \frac{1}{\mathbb{E}(R_H^{-1})} \mathbb{E}\left(f\left(\frac{1}{R_H}\right)\frac{1}{R_H}\right)$.

The identity in law $I \overset{\text{Law}}{=} e^{S_\infty} \times \dfrac{\widehat{I}_{\widehat{H}}}{R_H}$ reinforces the conjecture

$$\mathbf{P}(\log(I) > t) \sim c\,\mathbf{P}(\sup_{s>0} \xi_s > t), \qquad t \to \infty,$$

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Theorem

Assume ξ is no-monotone. The law of I has at least a power law (Pareto) tail:

$$\liminf_{t\to\infty} \frac{\log \mathbb{P}(I > t)}{\log t} > -\infty.$$

For $\alpha \geq 0$, the following are equivalent

- $t \mapsto \mathbb{P}(I > t)$ *is regularly varying at infinity with index* $-\alpha$,
- $t \mapsto \mathbb{P}(e^{S_\infty} > t)$ *is regularly varying at infinity with index* $-\alpha$.

In this case $\mathbb{P}(I > t) \sim \mathbb{E}\left(\widehat{I}^{\alpha}_{\widehat{H}} R_H^{-\alpha}\right) \mathbb{P}(e^{S_\infty} > t), \qquad t \to \infty.$

This is a consequence of results by Breiman (1965), Jacobsen, Mikosch, Rosinski, and Samorodnitsky (2009), and Goldie and Grübel (2000).

The identity $I \stackrel{\text{Law}}{=} \widehat{I}_{\widehat{H}} J_H$ leads to

Theorem

Assume that ξ has some negative jumps. For $\alpha \geq 0$, the following are equivalent

- *the function $t \mapsto \mathbb{P}(I \leq t)$ is regularly varying at 0 with index α*
- *the function $t \mapsto \mathbb{P}(\widehat{I}_{\widehat{H}} \leq t)$ is regularly varying at 0 with index α.*

In this case

$$\mathbb{P}(I \leq t) \sim \mathbb{E}\left(R_H^{\alpha-1}\right) \mathbb{P}(\widehat{I}_{\widehat{H}} \leq t), \qquad t \to 0.$$

According to Van Schaik, Pardo, R. 2012 a sufficient condition is that the left tail Lévy measure of ξ is in the class $\mathcal{L}_\alpha$

$$\lim_{x\to\infty} \frac{\overline{\Pi}^{-}(x+y)}{\overline{\Pi}^{-}(x)} = \exp\{-\alpha y\}, \quad y \in \mathbb{R},$$

with $\overline{\Pi}^{-}(x) = \Pi(-\infty, -x)$.

The potential measure $U(dy)$ is a renewal measure in the usual sense because

$$U(dy) = \sum_{n\geq 1} F^{*n}(dy), \qquad F(dy) := \mathbb{P}(\xi_{\mathbf{e}_1} \in dy),$$

with $\mathbf{e}_1$ an independent exponential r.v.
The new equation reads

$$\mathbb{P}(I > e^t) = \int_{\mathbb{R}} k(e^{t-y}) U(dy), \qquad t \in \mathbb{R}.$$

And thus one can use the artillery from renewal theory to study both $t \mapsto \mathbb{P}(I > e^t)$ and $t \mapsto \mathbb{P}(I \leq e^t)$.
Smith's renewal theorem ensures that when $\mathbb{E}(\xi_1) = m < 0$, then $\int_{\mathbb{R}} f(t-y)U(dy) \xrightarrow[t\to-\infty]{} \int_{\mathbb{R}} f(s)\frac{ds}{|m|}$, for directly Riemann Integrable functions f. Stone's decomposition gives information about the rate of convergence.

Using renewal theoretic arguments we can obtain results of the type

- Assume that ξ satisfies Cramér's condition:

$$\exists\gamma > 0, \quad \mathbb{E}(e^{\gamma\xi_1}) = 1, \qquad \mathbb{E}(\xi_1^m e^{\gamma\xi_1}) < \infty,$$

for some $m \geq 1$. Then,

$$\left| t^{\gamma}\,\mathbb{P}\left(I > t\right) - \frac{1}{\mathbb{E}(\xi_1 e^{\gamma\xi_1})}\,\mathbb{E}(I^{\gamma-1}) \right| = O\left(\frac{1}{\log^{m-1}(t)}\right).$$

If ξ is spectrally negative the order is $O(1/t)$.

Markov additive processes (MAPs)

- E is a finite state space
- $(J(t))_{t\geq 0}$ is a continuous-time, irreducible Markov chain on E
- the process (ξ, J) in $\mathbb{R} \times E$ is called a *Markov additive process (MAP)* with probabilities $\mathbb{P}_{x,i}$, $x \in \mathbb{R}, i \in E$, if, for any $i \in E$, $s, t \geq 0$: Given $\{J(t) = i\}$,
  - $(\xi(t+s) - \xi(t), J(t+s)) \perp \{(\xi(u), J(u)) : u \leq t\}$,
  - $(\xi(t+s) - \xi(t), J(t+s)) \stackrel{d}{=} (\xi(s), J(s))$ with $(\xi(0), J(0)) = (0, i)$.

Pathwise description of a MAP

The pair (ξ, J) is a Markov additive process if and only if, for each $i, j \in E$,

- there exist a sequence of iid Lévy processes $(\xi_i^n)_{n\geq 0}$
- and a sequence of iid random variables $(U_{ij}^n)_{n\geq 0}$, independent of the chain J,
- such that if $T_0 = 0$ and $(T_n)_{n\geq 1}$ are the jump times of J,

the process ξ has the representation

$$\xi(t) = 1_{\{n>0\}}(\xi(T_n-) + U^n_{J(T_n-),J(T_n)}) + \xi^n_{J(T_n)}(t - T_n),$$

for $t \in [T_n, T_{n+1})$, $n \geq 0$.
We are interested by the exponential functional of ξ

$$\int_0^\infty \exp\{\xi_s\} ds.$$

Characteristics of a MAP

- Denote the transition rate matrix of the chain J by $\mathbf{Q} = (q_{ij})_{i,j\in E}$.
- For each $i \in E$, the Laplace exponent of the Lévy process ξ_i will be written ψ_i (when it exists).
- For each pair of $i, j \in E$, define the Laplace transform $G_{ij}(z) = \mathbb{E}(e^{zU_{ij}})$ of the jump distribution U_{ij} (when it exists).
- Write $G(z)$ for the $N \times N$ matrix whose (i,j)th element is $G_{ij}(z)$.
- Let

$$\boldsymbol{\Psi}(z) = \mathrm{diag}(\psi_1(z), \cdots, \psi_N(z)) - \mathbf{Q} \circ G(z),$$

  (when it exists), where $\circ$ indicates elementwise multiplication.
- The matrix exponent of the MAP (ξ, J) is given by

$$\mathbb{E}_{0,i}(e^{z\xi(t)}; J(t) = j) = \left(e^{-\boldsymbol{\Psi}(z)t}\right)_{i,j}, \text{for } i,\ j \in E,$$

  (when it exists).

Lamperti-Kiu transform

- Take J to be irreducible on $E = \{1, -1\}$.

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- Let

$$X_t = |x|e^{\xi(\tau(|x|^{-\alpha}t))}J(\tau(|x|^{-\alpha}t)) \qquad 0 \leq t < T_0,$$

where

$$\tau(t) = \inf\left\{s > 0 : \int_0^s \exp(\alpha\xi(u))du > t\right\}$$

and

$$T_0 = |x|^{-\alpha}\int_0^\infty e^{\alpha\xi(u)}du.$$

- Then X_t is a real-valued self-similar Markov process in the sense that the law of $(cX_{tc^{-\alpha}} : t \geq 0)$ under $\mathbb{P}_x$ is $\mathbb{P}_{cx}$.

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and
$$T_0 = |x|^{-\alpha}\int_0^\infty \mathrm{e}^{\alpha\xi(u)}\mathrm{d}u.$$
- Then X_t is a real-valued self-similar Markov process in the sense that the law of $(cX_{tc^{-\alpha}} : t \ge 0)$ under $\mathbb{P}_x$ is $\mathbb{P}_{cx}$.
- The converse (within the special class of rssMps that die at its first hitting time of zero) is also true.

Theorem

Defining $\int_0^\infty \mathrm{e}^{\xi(u)}\mathrm{d}u$ *with* ξ *a Markov additive Lévy process as before, its probability distribution satisfies the equality of measures*

$$dt\,\mathbb{P}_i(I>t)=\sum_{j\in E}\int_{\mathbb{R}}U_{ij}(dy)e^y\,\mathbb{P}_j(e^yI\in dt),\qquad i\in E$$

$$U_{ij}(dy)=\mathbb{E}_{\xi_0=0,J(0)=i}\left(\int_0^\infty dt 1_{\{\xi_t\in dy,J_t=j\}}\right).$$

Further, if the probability density of I,

$$k_i(t)dt=\mathbb{P}_i(I\in dt),$$

exists for all $t>0$, *and* $i\in E$, *then we have*

$$\mathbb{P}_i(I>t)=\sum_{j\in E}\int_{\mathbb{R}}U_{ij}(dy)k_j(te^{-y}).$$

The (vectorial) recurrence relation holds $\mathbb{E}(I^z)=z\boldsymbol{\Psi}^{-1}(-iz)\,\mathbb{E}(I^{z-1})$.

Proposition

Let I be the exponential functional of the MAP ξ, and assume that ξ satisfies a Cramér's type condition with index $\theta > 0$, we have that

$$\mathbb{P}_i(I > t) \sim \nu_i t^{-\theta} \sum_{k \in E} \frac{\mathbb{E}_k(I^{\theta-1})}{\alpha \mu_\theta}, \text{ as } t \to \infty,$$

where $\mu_\theta = \mathbb{E}^{(\theta)}_{\pi^{(\theta)}} \xi_1$.

This results is proved as a consequence of the Markov renewal theorem.

In a work in progress with Andreas Kyprianou and Weerapat Satitkanitkul we are studying the questions

- What is the analogue of the residual exponential functional R?
- Is there a factorisation for I in the MAP case?

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Thank you very much for your attention!