#### Brownian trading excursions

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- In the limit order book (LOB), price level and number of orders away from the best bid & ask prices are recorded.
- We derive an SPDE for the relative order volume distribution which we can solve in terms of a local time functional.
- We will show that 'flash crashes' are important in the build up of the LOB, and study order execution avalanches.
- A bivariate Laplace-Mellin transform is derived for the joint height and length of flash crashes in a Brownian framework, involving the Riemann Xi-function.
- It is shown that the equality of two different representations of this transform, associated with two ways of disintegrating the Ito measure, is equivalent to the transformation formula of Jacobi's Theta function.

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### Limit order book model

- We work in business time and model the mid price process *W* as a Brownian motion; for a discussion of various price concepts in the context of the limit order book, see Delattre, Robert & Rosenbaum (2013). A more realistic model in real time would result from subordinating *B*.
- During an infinitesimal time interval dt, it is assumed that new limit orders are created at every level W<sub>t</sub> + u with volume density g(u) du, for some integrable function g : ℝ \ {0} → ℝ<sub>+</sub>.
- We can write the volume field in terms of order arrivals  $A_t^u$  as

$$V^u_t = \int_{\ell_u(t)}^t dA^u_s,$$

where  $\ell_u(t)$  denotes the *last exit time* from level *u* before time *t*, and

$$dA_t^u = g\left(u - W_t\right) \, dt.$$

### Basic model

• In our basic model, we consider the special case  $dA_t^u = \delta_{u-(W_t+\mu)} dt$ for some fixed displacement  $\mu > 0$ , and where  $\delta$  is the Dirac delta at level 0. This is just an informal specification; however, we can make it formal by setting

$$A_t^u = L_t^{u-\mu},$$

so the total volume of orders at level u (before any order gets executed) equals Brownian local time  $L_t^{u-\mu}$  at level  $u-\mu$ ; for a random walk this is just the number of times the walk hits this level. The total orders at some level get executed once the mid price hits this level.

 We will focus just on the ask side (sell orders), i.e. we assume g(x) = 0 for x < 0. Exogenous cancellations can be included in our model without much effort, but for the purpose of this talk will be excluded.

### SPDE for the order volume

• The relative volume random field  $v(t, x) := V(t, x + W_t)$  is the unique weak solution of the SPDE

$$dv(t,x) = \left(\frac{1}{2}\partial_x^2 v(t,x) + g\right) dt + \partial_x v(t,x) dW_t,$$
  
$$v(0,x) = g(x).$$

- Existence and uniqueness have been proved already in da Prato & Zabczyk (1992).
- We can express the solution in terms of local time as

$$v(t,x) = \int_{\mathbb{R}} \left( L_t^y - L_{\ell_x(t)}^y \right) g(x-y) \, dy + \mathbb{I}_{\tau_x=0} g(x).$$

where  $\ell_x(t)$  is the *last exit time* from level x before time t. A time reversed variant involving a first passage time is also possible.

• Let  $H := L^2((\mathbb{R}_+, \mathcal{B}, \lambda), \mathbb{R})$  and consider the spaces (all are dense in H)

 $\begin{aligned} H^1 &:= \left\{ f \in H : f \text{ is weakly differentiable and } f' \in H \right\}, \\ H^2 &:= \left\{ f \in H^1 : f' \in H^1 \right\}. \end{aligned}$ 

For any f ∈ H<sup>1</sup>, define the norm ||f||<sub>1</sub><sup>2</sup> := |f|<sup>2</sup> + |f'|<sup>2</sup> and denote the dual space by H<sup>-1</sup>. It contains for instance the point evaluations

$$\delta_{\mu}: H^1 \to \mathbb{R}, \ f \mapsto f(\mu).$$

• The weak derivative operator is denoted by

$$D: H^1 \to H, f \mapsto f'.$$

• We are looking for a weak solution  $v \in H^{-1}$  with given  $v(0) = g_0 \in H^{-1}$ ,  $g \in H^{-1}$  such that

$$d\mathbf{v}(t) = \left(\frac{1}{2}D^2\mathbf{v}(t) + g\right) dt + D\mathbf{v}(t) dW_t.$$

• Special case: Let  $A(t, x) := L_t^{x-\mu} - L_{\ell_x(t)}^{x-\mu}$  and  $a(t, x) := A(x + W_t)$ . Then a is the weak solution of the SPDE with a(0, x) = a(t, 0) = 0and  $g = \delta_\mu \in H^{-1}$  in the sense that for all  $\phi \in H_0^2$ , we have

$$d\left\langle \phi, \mathsf{a}(t) 
ight
angle = \left(rac{1}{2}\left\langle D_{0}^{2}\phi, u(t) 
ight
angle + \phi\left(\mu
ight)
ight) \, dt + \left\langle D\phi, \mathsf{a}(t) 
ight
angle \, dW_{t}.$$

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### Execution mechanisms

- As a special case, we consider g(x) = δ<sub>x+μ</sub> for some fixed displacement μ > 0, so the total volume of orders at level u equals the local time at level u μ. We now assume that the Brownian motion W, W<sub>0</sub> = 0, has just surpassed the level μ:
- Type I execution: an order is triggered whenever the running maximum of B is increasing.
- O Type II execution: like type I, but after a downward excursion straddling at least a size of μ. If the excursion lasts less than time ε we call it a 'flash crash'.
  - We take record if there is no order execution in a time period lasting longer than ε, i.e. when a downward excursion takes place with duration at least of ε.

- We say that a random time  $\tau > 0$  is a *trading time* if  $V(\tau_{-}, W_{\tau}) > 0$ and  $V(\tau, W_{\tau}) = 0$ .
- Let Y (τ) denote the supremum of trading times before trading time τ; if there was no trading time before τ we set Y (τ) = 0. Similarly, let Ξ (τ) denote the infimum of trading times after trading time τ.
- We say that a *type I* trade occurs if either  $Y(\tau) = \tau$  and  $\tau = \Xi(\tau)$ (*type la* trade) or  $Y(\tau) = \tau$  but  $\tau \neq \Xi(\tau)$  (*type Ib* trade) or  $Y(\tau) \neq \tau$ , but  $\sup_{Y(\tau) \leq t \leq \tau} (W_{Y(\tau)} - W_t) < \mu$  (*type Ic* trade).
- Otherwise we call it a *type II* trade.

- Orders in the LOB get executed via avalanches. In other words, limit orders may accumulate on some levels, and when the price process crosses those values, we will see a sudden decrease of the number of orders in the LOB.
- Let  $T_{\varepsilon}^{\text{start}}$  denote the time when there is the first order execution after a time of at least  $\varepsilon$  since the last execution, and  $T_{\varepsilon}^{\text{end}}$  similarly the last execution time before a downward excursion lasting at least  $\varepsilon$ takes place.
- An  $\varepsilon$ -avalanche is defined as the stopped process  $\{W_t : T_{\varepsilon}^{\text{start}} \leq t \leq L^{\varepsilon}\}$  where the avalanche length  $L^{\varepsilon}$  is the difference between the last and the first execution time,

$$L^{\varepsilon} := T_{\varepsilon}^{\text{end}} - T_{\varepsilon}^{\text{start}}.$$

• We are interested into the distribution of the avalanche length.

- Let us first assume that there is no downward excursion straddling at least a size of  $\mu$ , and lasting less than time  $\varepsilon$ .
- Dassios and Lim (2009) derive the Laplace transform of the avalanche length  $L^{\varepsilon}$  in the context of Parisian options as

$$E\left[e^{-\lambda L^{arepsilon}}
ight]=rac{1}{\sqrt{\lambdaarepsilon\pi}\,\mathrm{erf}\left(\sqrt{\lambdaarepsilon}
ight)+e^{-\lambdaarepsilon}}$$

• The same formula can be inferred (Laurent de Dok du Wit, Diploma Thesis 2012) from the Lévy measure of the subordinator consisting of Brownian passage times.

### Hyperbolic function table for intertrading times

• Hyperbolic function table. Assume  $\tau$  is a type Ib trade. Let T denote the time to the next trade, S the time to the next type Ic trade, U the time to the next type II trade. Finally, let  $\sigma$  be any trading time, and denote by U' the time to the next type II trade. Then we get for the Laplace transforms

$$E\left[e^{-\lambda T}\right] = 1 - \mu \sqrt{2\lambda} \tanh(\mu \sqrt{2\lambda}); \qquad (1)$$

$$E\left[e^{-\lambda S}\right] = 1 - \mu \sqrt{2\lambda} \coth(2\mu \sqrt{2\lambda}); \qquad (2)$$

$$E\left[e^{-\lambda U}\right] = \mu \sqrt{2\lambda} \csc h(2\mu \sqrt{2\lambda}); \tag{3}$$

$$E\left[e^{-\lambda U'}\right] = \mu \operatorname{sec} h(2\mu\sqrt{2\lambda})^2.$$
(4)

- Regarding entry (3) of the hyperbolic table, we are interested in the time U to the next Type II trade, conditioned that we have no Type I trade before. This means that we condition on that there will be, starting from some fixed level x = W(τ), a downward excursion with height of at least μ. Denote the first hitting time of the level x μ by T<sub>μ</sub>. The process x W on [τ, T<sub>μ</sub>] has then the law of a three-dimensional Bessel process BES<sub>3</sub>. By Biane, Pitman & Yor (2001) the Laplace transform of T<sub>μ</sub> is μ√2λ/sinh(μ√2λ). The significance of T<sub>μ</sub> is that after this time, it is certain that there will be a Type II trade before the next Type I trade.
- As  $T_{\mu}$  is a stopping time for the filtration generated by W, after  $T_{\mu}$  by the strong Markov property W has the law of a Brownian motion. The Type II trade will get triggered once  $W_t \min_{t \ge T_{\mu}} W_t$  equals  $\mu$ , for  $t \ge T_{\mu}$ . We have that  $W \min W$  has the same law as |W|. The Laplace transform of the modulus of Brownian motion equals  $1/\cosh\left(\mu\sqrt{2\lambda}\right)$ . Using the doubling formula for the hyperbolic sine, it results that

$$E\left[e^{-\lambda U}\right] = \mu \sqrt{2\lambda} \operatorname{csc} h\left(2\mu \sqrt{2\lambda}\right)$$

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# Laplace transform for full avalanche length, including flash crashes

• We can show that the Laplace transform of the full avalanche length  $A^{\varepsilon}$  is given as

$$E\left[e^{-\lambda A^{\varepsilon}}\right] = \frac{\int_{\varepsilon}^{\infty} h(x) \, dx}{\int_{0}^{\varepsilon} \left(1 - e^{-\lambda x}\right) h(x) \, dx + \int_{\varepsilon}^{\infty} h(x) \, dx}$$

where

$$h(x) = \frac{x^{-3/2}}{\sqrt{2\pi}} + 2\sum_{k\geq 1} \left(\frac{x^{-3/2}}{\sqrt{2\pi}} - 2\sqrt{\frac{2}{\pi}}\frac{k^2y^2}{x^{5/2}}\right) e^{-2k^2y^2/x}$$

and

$$\int_0^\infty (1-e^{-\lambda x})h(x)dx = \sqrt{2\lambda} \tanh\left(\mu\sqrt{2\lambda}\right).$$

### A short primer on excursion theory

- Denote by (U, U) the measurable space of Brownian excursions, and by (e<sub>t</sub>, t > 0) the excursion process. We enhance it by the zero-excursion (which is set equal to δ) on the set where the local time at zero is strictly increasing, and denote the resulting space by U<sub>δ</sub> = U ∪ {δ}, equipped with the σ-algebra U<sub>δ</sub> = σ (U, {δ}).
- For a measurable subset  $\Gamma$  of  $\mathcal{U}_{\delta}$ , one sets

$$N_{t}^{\Gamma}(\omega) = \sum_{0 < u \leq t} \mathbf{1}_{\Gamma}(e_{u}(\omega)).$$

• The *Ito measure* n is the  $\sigma$ -finite measure defined on  $\mathcal U$  by

$$n\left(\Gamma\right):=E\left[N_{1}^{\Gamma}\right]$$

and extended to  $\mathcal{U}_{\delta}$  by  $n\left(\delta
ight)=$  0.

• It turns out that the excursion process is a Poisson Point Process, and hence the Ito measure is its characteristic measure.

• Denoting by *R* the excursion length, the density of *R* under *n*<sup>+</sup> (the Ito measure restricted to positive excursions) is

$$\frac{1}{2\sqrt{2\pi r^3}}$$

- Moreover, under n<sup>+</sup> and conditionally on R = r, the coordinate process w has the law π<sub>r</sub> of the Bessel Bridge of dimension 3 over [0, r].
- Hence if  $\Gamma$  is a measurable subset of  $U^+$ , then

$$n_{+}(\Gamma) = \int_{0}^{\infty} \pi_{r} \left( \Gamma \cap \{ R = r \} \right) \frac{dr}{2\sqrt{2\pi r^{3}}}.$$

### Moment generating function of normalized excursion height

• In fact, the mgf of the height N of the normalized Brownian excursion can be determined as

$$\frac{1}{2}E\left[\left(\frac{\pi}{2}N\right)^{s}\right] = \xi(2s), \quad s > 1,$$

where  $\xi$  denotes the Riemann Xi function which is connected to the Riemann zeta function  $\zeta$  by

$$\xi\left(s
ight)=rac{s\left(s-1
ight)}{2}\Gamma\left(rac{s}{2}
ight)\pi^{-s/2}\zeta\left(s
ight).$$

• In particular, the Xi function has no zeroes outside of the critical strip, and satisfies the reflection principle

$$\xi\left(1-s
ight)=\xi\left(s
ight).$$

• Our main result gives a transform of the joint distribution of the excursion length R and its height H. This transform is a bivariate Laplace-Mellin transform and states that for  $\lambda > 0$ , s > 1

$$\int_{U} e^{-\lambda R(u)} H(u)^{s-1} n^{+}(du) = \frac{1}{\sqrt{8\pi}} \lambda^{1-\frac{s}{2}} \Gamma\left(\frac{s}{2}-1\right) \xi\left(\frac{s}{2}\right).$$

• This allows us to determine the joint law

$$n^+$$
 ( $R < \varepsilon; H > \mu$ ),

i.e. the distribution of a flash crash under the Ito measure.

### Two differerent descriptions of the Ito measure

• The joint density  $f_{H,R}$  of length R and height H under n can be written as, see Pollard (2002) for a rigorous treatment about conditioning,

 $f_{H,R}(h, r) = f_{H|R}(h, r)f_R(r)$  Ito – description

or

 $f_{H,R}(h, r) = f_{R|H}(h, r)f_H(h)$  Williams – description

- Ito's description involves the distribution of the height of a three-dimensional Bessel bridge.
- Williams' description is based on the distribution of the sum of two independent copies of the hitting time of 1 by a three-dimensional Bessel process starting in zero.
- Whereas Ito's description involves the distribution of the height of a three-dimensional Bessel bridge, Williams' description is based on the distribution of the sum of two independent copies of the hitting time of 1 by a three-dimensional Bessel process starting in zero.

### Equality of two disintegrations

• These two different disintegrations are equivalent, but the densities involved have different expressions. These leads for instance to the equality, valid for all h, r > 0,

$$\frac{\pi^2}{h^4} \sum_{n=1}^{\infty} e^{-\frac{m^2 \pi^2}{2h^2}} \left( \frac{rn^4 \pi^2}{2h^2} - \frac{3n^2}{2} \right) = \frac{h\sqrt{32}}{\sqrt{\pi r^5}} \sum_{n=1}^{\infty} e^{-\frac{2h^2 n^2}{r}} \left( \frac{2h^2 n^4}{r} - \frac{3n^2}{2} \right),$$

which is a formula encoding the equivalence of Ito's and William's description.

- There is an interesting connection to Jacobi's Theta function which is defined for  $\tau > 0$ , and normally convergent for  $\tau > \varepsilon > 0$ , by  $\vartheta(\tau) = 1 + 2\sum_{n=1}^{\infty} e^{-\pi n^2 \tau}$ .
- It satisfies the transformation formula, for au > 0,

$$artheta\left(1/ au
ight)=\sqrt{ au}artheta\left( au
ight)$$
 .

• We can show that the equality between Ito's and Williams' disintegrations of the Ito measure is equivalent to the transformation formula of Jacobi's Theta function, see also Mansuy & Yor (2008).

### Thank You for Your attention!