The dichotomy of recurrence and transience of semi-Lévy processes

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Additive processes

Definition

(1) $\{X_t, t \geq 0\}$ on $\mathbb{R}^d$ is an *additive process*, if

(i) $X_0 = 0$ a.s.

(ii) It has independent increments.

(iii) It is stochastically continuous at every $t \geq 0$.

(iv) For almost all ω, the sample paths $X_t(\omega)$ are right-continuous in $t \geq 0$ and have left limits in $t > 0$.

(2) $\{X_t, t \geq 0\}$ on $\mathbb{R}^d$ is a *Lévy process*, if (i)-(iv) and

(v) it has stationary increments, namely

$$X_{t+h} - X_{s+h} \overset{\mathrm{d}}{=} X_t - X_s, \quad \forall t, s, \geq 0, \ \forall h \geq 0.$$

Recurrence and transience for additive processes

Definition

Fix $s \geq 0$. An additive process $\{X_t, t \geq 0\}$ on $\mathbb{R}^d$ is called *s-recurrent* if

$$\liminf_{t \to \infty} |X_t - X_s| = 0 \quad \text{a.s.}, \tag{2.1}$$

and it is called *recurrent* if it is s-recurrent for any $s \geq 0$.
It is called *transient* if

$$\lim_{t \to \infty} |X_t| = \infty \quad \text{a.s.}$$

Remark

Note that, for Lévy processes, by the stationary increment property, recurrence and 0-recurrence are equivalent.
Thus, (2.1) can be replaced by

$$\liminf_{t \to \infty} |X_t| = 0 \quad \text{a.s.},$$

The dichotomy of recurrence and transience

Theorem (The dichotomy for Lévy processes (Kingmann (1964).)

A Lévy process is recurrent or transient. (= *The dichotomy holds.*)

Example

(An additive process not having the dichotomy property.)
(Sato-Yamamuro (1998).)
$\{X_t\}$: non-zero Lévy process
$a(t)$: strictly increasing continuous function with $a(0) = 0$ and $a(\infty) < \infty$
Let $Y_t := X_{a(t)}$.
$\Longrightarrow$ (1) $\{Y_t\}$ is an additive process.
(2) $Y_t \to X_{a(\infty)-}$ as $t \to \infty$.
(Therefore, $\{Y_t\}$ is neither recurrent nor transient.)

Examples

(Additive processes having the dichotomy property)

(1) Lévy process

(2) Selfsimilar additive process (Sato process) (by Sato-Yamamuro (1998))

(3) Semi-selfsimilar additive process (by Sato-Yamamuro (1998))

(Semi-selfsimilar process $\stackrel{\text{def}}{\Longleftrightarrow} \exists a > 0, \exists H > 0$ s.t. $\{X_{at}\} \stackrel{\text{d}}{=} \{a^H X_t\}$)

Remark

(1) and (2) are semimartingale, but (3) is not necessarily semimartingale.

Semi-Lévy processes

Definition

(M.-Sato (2003))
An additive process $\{X_t, t \geq 0\}$ is a semi-Lévy process if
(v') (having periodically stationary increments)

$$\exists p > 0 \text{ s.t. } X_{t+p} - X_{s+p} \stackrel{\mathrm{d}}{=} X_t - X_s \quad \forall\ s, t \geq 0.$$

p is called a period of the semi-Lévy process $\{X_t\}$.

The background of introducing the concept of semi-Lévy process

(1) $\mu \in ID(\mathbb{R}^d)$ is selfdecomposable
$\overset{\text{def}}{\Longleftrightarrow} \forall b > 1, \exists \mu_b \in ID(\mathbb{R}^d)$ s.t. $\widehat{\mu}(z) = \widehat{\mu}(b^{-1}z)\widehat{\mu}_b(z)$
$\overset{\text{iff}}{\Longleftrightarrow} \exists$ a Lévy process $\{X_t\}$ s.t. $E[\log(1 + |X_1|) < \infty, \mu = \mathcal{L}\left(\int_0^\infty e^{-t} dX_t\right)$
(2) $\mu \in ID(\mathbb{R}^d)$ is semi-selfdecomposable with span $b > 1$
$\overset{\text{def}}{\Longleftrightarrow} \exists \rho \in ID(\mathbb{R}^d)$ s.t. $\widehat{\mu}(z) = \widehat{\mu}(b^{-1}z)\widehat{\rho}(z)$
$\overset{?}{\Longrightarrow}$ stochastic integral representation?
(3) (Properties of a semi-Lévy process $\{X_t\}$)

(i) It is not necessarily semimartingale.

(ii) (Sato (2004)) An additive process $\{X_t\}$ is called *natural* if the scaling part γ_t of the Lévy-Khintchine representation of $\mathcal{L}(X_t)$ is of locally bounded variation. For an additive process $\{X_t\}$, it is natural iff it is semimartingale.

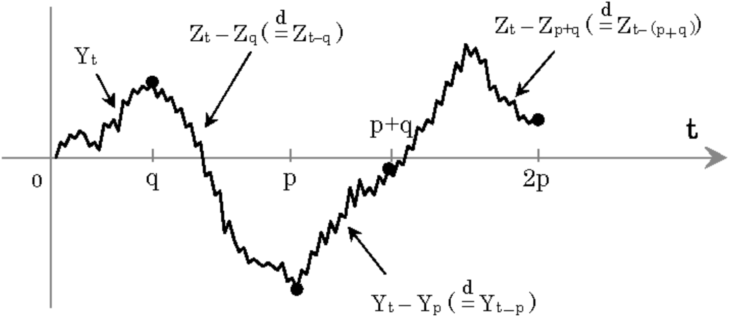
(4) (M.-Sato (2003))
$\mu \in ID(\mathbb{R}^d)$ is semi-selfdecomposable with span $b > 1$
$\overset{\text{iff}}{\Longleftrightarrow}$
$\exists$ a natural semi-Lévy process $\{X_t\}$ with period $p = \log b$
s.t. $E[\log(1 + |X_p|) < \infty, \mu = \mathcal{L}\left(\int_0^\infty e^{-t} dX_t\right)$

An example of semi-Lévy processes (by K. Sato)

Let $\{Y_t, t \geq 0\}$ and $\{Z_t, t \geq 0\}$ be two independent Lévy processes and let $0 < q < p$ be arbitrary. A stochastic process $\{X_t, t \geq 0\}$ defined by

$$
\begin{aligned}
&X_0 = 0 \quad \text{a.s.},\\
&X_t = \begin{cases} X_{np} + Y_t - Y_{np} & (np < t \leq np + q),\\ X_{np+q} + Z_t - Z_{np+q} & (np + q < t \leq (n+1)p) \end{cases}
\end{aligned}
$$

is a semi-Lévy process with period p. The recurrence and transience property of this process will be discussed at the end of this talk.



$Z_t - Z_q \,(\stackrel{d}{=} Z_{t-q})$
$Z_t - Z_{p+q} \,(\stackrel{d}{=} Z_{t-(p+q)})$
Y_t
$p+q$
t
0
q
p
2p
$Y_t - Y_p \,(\stackrel{d}{=} Y_{t-p})$

Semi-random walks

Definition

Fix $p \in \mathbb{N}$. $\{S_n, n \in \mathbb{Z}_+\}$ (on $\mathbb{R}^d$) is a semi-random walk with period $p \in \mathbb{Z}_+$

$\overset{\text{def}}{\iff}$

(i) $S_0 = 0$ a.s.

(ii) $S_n - S_m$ is independent of $S_k - S_\ell$,

$$\forall n, m, k, \ell \in Z_+, n > m \geq k > \ell,$$

(iii) $S_{n+p} - S_{m+p} \overset{\mathrm{d}}{=} S_n - S_m, \quad n, m \in \mathbb{Z}_+$

Remark

$\{S_n, n \in \mathbb{Z}_+\}$: a semi-random walk with period p

$\Longrightarrow \{S_{np}, n \in \mathbb{Z}_+\}$: a random walk

Definition

(1) Fix $m \in \mathbb{Z}_+$. A semi-random walk $\{S_n\}$ is m-recurrent

$$\overset{\text{def}}{\Longleftrightarrow} \liminf_{n\to\infty} |S_n - S_m| = 0 \text{ a.s.}$$

(2) $\{S_n\}$ is recurrent $\overset{\text{def}}{\Longleftrightarrow}$ it is m-recurrent for $\forall m \in \mathbb{Z}_+$.

(3) $\{S_n\}$ is transient $\overset{\text{def}}{\Longleftrightarrow}$ $\lim_{n\to\infty} |S_n| = \infty$ a.s.

Example

(An example of semi-random walks)

$\{X_t\}$: semi-Lévy process with period $p(>0)$.

$h \in (0,\infty) \cap p\mathbb{Q}$, i.e. $h = p\frac{n_1}{n_2}, n_1, n_2 > 0$.

$\Longrightarrow \{X_{nh}, n \in \mathbb{Z}_+\}$: semi-random walk with period $n_2 \in \mathbb{N}$.

(*Proof*) $\forall n, m \in \mathbb{Z}_+$,

$$X_{(n+n_2)h} - X_{(m+n_2)h} = X_{nh+pn_1} - X_{mh+pn_1} \overset{\mathrm{d}}{=} X_{nh} - X_{mh}$$

The dichotomy of recurrence and transience of semi-Lévy processes

The answer is YES!

(M.-Takamune-Ueda, *J. Theor. Probab.* **27** (2014), 982-996.)

Lemma (A)

Let $\{X_t\}_{t\geq 0}$ *be a semi-Lévy process on* $\mathbb{R}^d$ *with period* $p > 0$.
The following are equivalent.
(i) *It is* 0*-recurrent.*
(ii)

$$\int_0^\infty \mathbb{1}_{B_a}(X_t)dt = \infty \quad \textit{a.s.} \quad \textit{for every } a > 0.$$

(*Proof*) For the proof, we need the following statement on semi-random walk.
(iii) There exists $h_0 > 0$ such that, for any $h \in (0, h_0] \cap p\mathbb{Q}$, the semi-random walk $\{X_{nh}\}_{n\in\mathbb{Z}_+}$ is recurrent.
We prove the equivalence of (i) and (ii), by showing that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i).

Theorem (A)

Let $\{X_t\}_{t\geq 0}$ *be a semi-Lévy process on* $\mathbb{R}^d$ *with period* $p > 0$.
It is transient if and only if

$$\int_0^\infty P(X_t \in B_a)dt < \infty \quad \textit{for every } a > 0.$$

(For the proof, we use Lemma (A).)

Theorem (B)

Let $\{X_t\}_{t\geq 0}$ *be a semi-Lévy process on* $\mathbb{R}^d$ *with period* $p > 0$.
(i) *If it is* 0*-recurrent, then it is recurrent.*
(ii) *It is recurrent if and only if*

$$\int_0^\infty P(X_t \in B_a)dt = \infty \quad \textit{for every } a > 0.$$

(iii) *It is either recurrent or transient.*

For the proof, we use Lemma (A) and Theorem (A).

Further, let $\{X_t\}_{t\geq 0}$ be a semi-Lévy process with period $p > 0$. Since X_p is an infinitely divisible random variable, then there is a Lévy process $\{Y_t\}_{t\geq 0}$ such that $Y_1 \stackrel{\mathrm{d}}{=} X_p$. Now, as a consequence of the previous Theorem , we have the following.

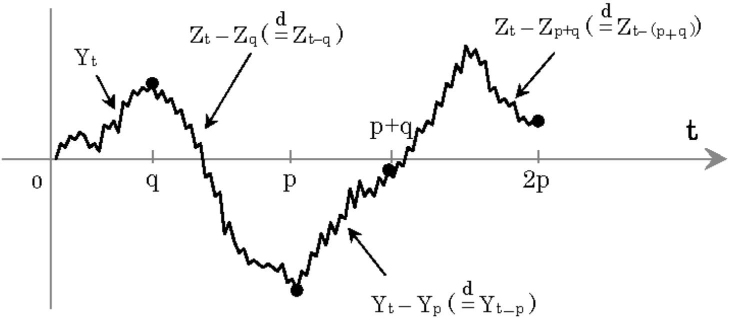
Theorem (C)

The semi-Lévy process $\{X_t\}_{t\geq 0}$ is recurrent if and only if the Lévy process $\{Y_t\}_{t\geq 0}$ is recurrent.

For the proof, we use the following statement on semi-random walk. Fix $h \in (0, \infty) \cap p\mathbb{Q}$ arbitrarily. Then, the semi-Lévy process $\{X_t\}_{t\geq 0}$ is recurrent if and only if the semi-random walk $\{X_{nh}\}_{n\in\mathbb{Z}_+}$ is recurrent.

Example

Let $\{Y_t\}_{t\geq 0}$ and $\{Z_t\}_{t\geq 0}$ be two independent Lévy processes on $\mathbb{R}^d$ and let $0 < q < p$ be arbitrary. $\{X_t\}_{t\geq 0}$ defined by



Proposition

Assume $E[|Y_1|] < \infty$ *and* $E[|Z_1|] < \infty$.
Then $\{X_t\}_{t\geq 0}$ *is recurrent* $\overset{\text{iff}}{\Longleftrightarrow} E[Y_q + Z_{p-q}] = 0$.

Corollary

Assume $E[|Y_1|] < \infty$ *and* $E[|Z_1|] < \infty$.
(i) $\{Y_t\}_{t\geq 0}$ *and* $\{Z_t\}_{t\geq 0}$ *are recurrent* $\Longleftrightarrow E[Y_1] = E[Z_1] = 0 \Longrightarrow E[Y_q + Z_{p-q}] = 0 \Longleftrightarrow$ $\{X_t\}_{t\geq 0}$ *is recurrent.*
(ii) $\{Y_t\}_{t\geq 0}$ *is recurrent and* $\{Z_t\}_{t\geq 0}$ *is transient* $\Longleftrightarrow E[Y_1] = 0$ *and* $E[Z_1] \neq 0 \Rightarrow E[Y_q + Z_{p-q}] \neq 0 \Longleftrightarrow$ $\{X_t\}_{t\geq 0}$ *is transient.*
(iii) $\{Y_t\}_{t\geq 0}$ *and* $\{Z_t\}_{t\geq 0}$ *are transient* $\Longrightarrow \{X_t\}_{t\geq 0}$ *is recurrent if* $E[Y_q + Z_{p-q}] = 0$ *and it is transient if* $E[Y_q + Z_{p-q}] \neq 0$.

Remark

Let us remark that in the case when $E[|Y_1|] = \infty$, (ii) of Corollary does not have to hold. For example, let $\{Y_t\}_{t\geq 0}$ be a symmetric Cauchy process and let $\{Z_t\}_{t\geq 0}$ be an arbitrary Lévy process with $E[|Z_1|] < \infty$. Recall that $\{Y_t\}_{t\geq 0}$ is recurrent (see Example 35.7 of Sato (1999)) and $E[|Y_1|] = \infty$. Now, by Exercise 39.8 (i) and (iv) of Sato (1999) and Proposition 4.1, $\{X_t\}_{t\geq 0}$ is recurrent.