The Probability Density Function of Turbulence

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decicated to Ole Barndorff-Nielsen on his 80th birthday

Outline

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The Deterministic Navier-Stokes Equations

- A general incompressible fluid flow satisfies the Navier-Stokes Equation

$$\begin{aligned} u_t + u \cdot \nabla u &= \nu \Delta u - \nabla p \\ u(x,0) &= u_0(x) \end{aligned}$$

with the incompressibility condition

$$\nabla \cdot u = 0,$$

- Eliminating the pressure using the incompressibility condition gives

$$\begin{aligned} u_t + u \cdot \nabla u &= \nu \Delta u + \nabla \Delta^{-1} \mathrm{trace}(\nabla u)^2 \\ u(x,0) &= u_0(x) \end{aligned}$$

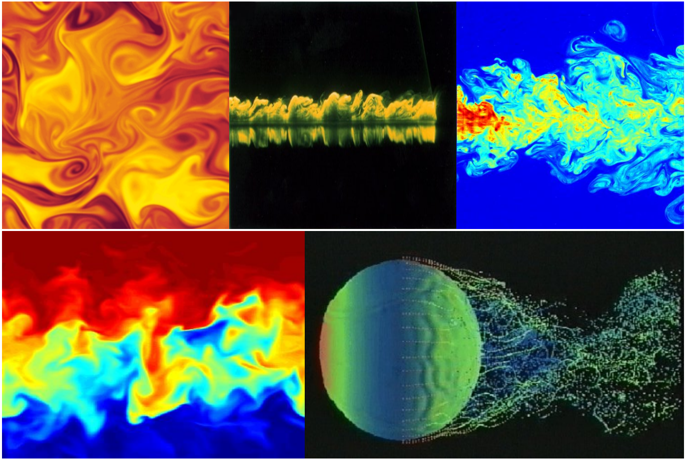
- The turbulence is quantified by the dimensionless Taylor-Reynolds number $Re_\lambda = \frac{U\lambda}{\nu}$

Laminar and Turbulent Flow

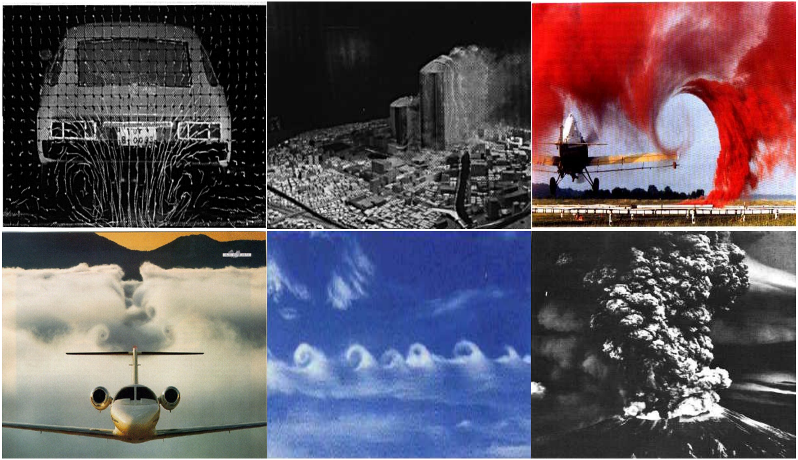
The Reynolds Number : $Re = \frac{UL}{\nu}$

- In 1883 the mechanical engineer Osborne Reynolds observed:
  "The internal motion of water assumes one or other of two broadly distinguishable forms-either the elements of the fluid follow one another along lines of motion which lead in the most direct manner to their destination or they eddy about in sinuous paths the most indirect possible."
- These are respectively laminar and turbulent flow

Images of Turbulence



Turbulence in Applications



The Reynolds Decomposition

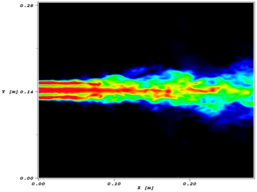
- The velocity is written as $U+u$, pressure as $P+p$ U describes the large scale flow, u describes the small scale turbulence
- This is the classical Reynolds decomposition (RANS)

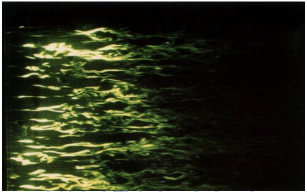
$$U_t + U \cdot \nabla U \quad = \quad \nu \Delta U - \nabla P - \frac{\partial}{\partial x_j} \mathcal{R}_{ij}$$

- The last term the eddy viscosity, where $\mathcal{R}_{ij} = \overline{u_i u_j}$ is the Reynolds stress, describes how the small scale influence the large ones. *Closure problem*: compute $\mathcal{R}_{ij}$.

The Difference between Laminar and Turbulent

- If the Reynolds number is small, only the laminar solution exists
  In this case the ambient noise is quelled
- If the Reynolds number is large, the laminar solution exists but is unstable
  The ambient noise is magnified by the instabilities of the laminar flow and becomes large
- Then the turbulent solution satisfies a stochastic partial differential equation (SPDE)





A Stochastic Closure

- Large scale flow

$$\begin{aligned} U_t + U \cdot \nabla U &= \nu \Delta U - \nabla P - \frac{\partial}{\partial x_j} \mathcal{R}_{ij} \\ U(x,0) &= U_o(x). \end{aligned}$$

- Small scale flow

$$\begin{aligned} u_t + u \cdot \nabla u &= \nu \Delta u + \nabla \Delta^{-1} \mathrm{trace}(\nabla u)^2 + \mathrm{Noise} \\ u(x,0) &= u_0(x). \end{aligned}$$

- What is the form of the Noise? It will contain both additive noise and multiplicative $u \cdot$ noise.

The Central Limit Theorem

- Split the torus $\mathbb{T}^3$ into little boxes and consider the dissipation to be a stochastic process in each box
  By the central limit theorem the scaled average

$$M_n = \frac{1}{n}\sum_{j=1}^{n} p_j$$

  $\sqrt{n}(M_n - m)/\sigma \to N(0,1)$ converges to a Gaussian (normal) random variable as $n \to \infty$
- This holds for any Fourier component (e_k) and the result is the infinite dimensional Brownian motion

$$df_t^{(1)} = \sum_{k \neq 0} c_k^{\frac{1}{2}} db_t^k e_k(x)$$

Fluctuations and the large deviation principle

- In addition we get fluctuation in the mean of the dissipation
  If these fluctuation are completely random then they are modeled by Poisson process with the rate μ
- Applying the large deviation principle, get a deterministic bound, with rate μ_k
- This also holds in the direction of each Fourier component and gives Fourier series

$$df_t^{(2)} = \sum_{k \neq 0} d_k \mu_k dt\ e_k(x), \quad \mu_k = |k|^{1/3}$$

Intermittency and velocity fluctuation

- The multiplicative noise, models the excursion (jumps) in the velocity gradient (vorticity concentrations)
- N_t^k denotes the integer number of velocity excursion, associated with kth wavenumber, that have occurred at time t. The differential $dN^k(t) = N^k(t+dt) - N^k(t)$ denotes these excursions in the time interval $(t, t+dt]$.
- The process

$$df_t^{(3)} = \sum_{k \neq 0}^{M} \int_{\mathbb{R}} h_k(t,z) \bar{N}^k(dt,dz),$$

gives the multiplicative noise term

Stochastic Navier-Stokes with Turbulent Noise

- Adding the two types of additive noise and the multiplicative noise we get the stochastic Navier-Stokes equations describing fully developed turbulence

$$
\begin{aligned}
du = (\nu\Delta u &- (U+u)\cdot\nabla u - u\cdot\nabla U + \nabla\Delta^{-1}\mathrm{tr}(\nabla u)^2)dt \\
&+ \sum_{k\neq 0} c_k^{\frac{1}{2}} db_t^k e_k(x) + \sum_{k\neq 0} d_k|k|^{1/3}dt\ e_k(x) \\
&+ u(\sum_{k\neq 0}^{M} \int_{\mathbb{R}} h_k \bar{N}^k(dt,dz)) \qquad (1) \\
u(x,0) &= u_0(x)
\end{aligned}
$$

- Each Fourier component $e_k = e^{2\pi i k\cdot x}$ comes with its own Brownian motion b_t^k and deterministic bound $|k|^{1/3}dt$

Outline

The Kolmogorov-Obukhov Theory

- In 1941 Kolmogorov and Obukhov [10, 9, 16] proposed a statistical theory of turbulence
- The structure functions of the velocity differences of a turbulent fluid, should scale with the distance (lag variable) l between them, to the power $p/3$

$$E(|u(x,t)-u(x+l,t)|^p) = S_p = C_p l^{p/3}$$


A. Kolmogorov

A. Obukhov

The Kolmogorov-Obukhov Refinded Similarity with She-Leveque Intermittency Corrections

- The Kolmogorov-Obukhov '41 theory was criticized by Landau for including universal constants C_p and later for not including the influence of the intermittency
- In 1962 Kolmogorov and Obukhov [11, 17] proposed a refined similarity hypothesis

$$S_p = C_p' < \tilde{\varepsilon}^{p/3} > l^{p/3} = C_p l^{\zeta_p} \tag{2}$$

l is the lag and ε a mean energy dissipation rate

- The scaling exponents

$$\zeta_p = \frac{p}{3} + \tau_p$$

include the She-Leveque intermittency corrections $\tau_p = -\frac{2p}{9} + 2(1-(2/3)^{p/3})$ and the C_p are not universal but depend on the large flow structure

Why do we need a Statistical Theory of Turbulence?

- Kolmogorov's point of view was that the fluid velocity in turbulence was not a deterministic function but rather a stochastic process
- The reason for this was, that one had to solve the Navier-Stokes equation in a noisy environment to obtain the velocity. This noise had been created by the fluid instabilities magnifying ambient noise. Once the noise was present it could not be ignored
- The consequence is that the only deterministic quantities associated with the turbulent velocity are statistical quantities such as the mean, the variance, the skewness, the kurtosis and so on. We must use probability theory or statistics to study turbulence

Experimental observation of the Kolmogorov-Obukhov scaling $E(k) = C\varepsilon_0^{2/3}k^{-5/3}$

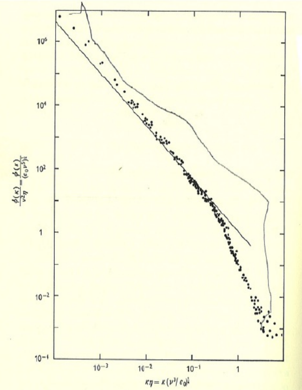


Fig. 6.2. The turbulence spectra, measured by Grant, Stewart and Moilliet (1962) and scaled according to the Kolmogorov parameters. The viscous dissipation rate ϵ_0 varied over a range of values of the order 100. The straight line represents variation as $\kappa^{-\frac{5}{3}}$. The top few points are believed to be rather high on account of the low frequency heaving motions of the ship.

Figure: The first convincing data was obtained by Grant et al. in 1962, on the turbulence in a tidal stream in the Seymour Narrows, between Vancouver and Quadra Islands, British Columbia.

The Kolmogorov-Obukhov Theory

The (kinetic) energy in the turbulent velocity field is

$$\mathcal{E}(t) = \frac{1}{2}\int_{\mathbb{R}^3} |u(x,t)|^2 dx = \frac{1}{2}\int_{\mathbb{R}^3} |\hat{u}(k,t)|^2 dk$$

using Plancherel's identity. The last integral can be written as

$$\int_0^\infty |k|^2 \int_{S^2_{|k|}} |\hat{u}(k,t)|^2 d\omega d|k| = \int_0^\infty |k|^2 E(k,t) d|k|$$

using polar coordinates, where

$$E(k,t) = \frac{1}{2}\int_{S^2_{|k|}} |\hat{u}(k,t)|^2 d\omega$$

is the integral over a two-dimensional sphere (spherical shell) of radius $|k|$.

The First Hypothesis

There exists a statistical equilibrium

- Thus $E(k,t)$ is the energy density per wave number k, independent of direction (isotropic). From the Navier-Stokes equations, we get that:

$$\frac{\partial E}{\partial t}(k) = -\frac{\partial}{\partial k}\varepsilon(k) - 2\nu k^2 E(k) + \cdots$$

where $k = |k|$. $\varepsilon(k)$ is the spectral dissipation rate or the rate of transfer of energy from the wavenumers less that k to the wave numbers larger than k.

- The first Hypothesis say that there exists a statistical stationary state, for sufficiently high Reynolds numbers,

$$\frac{\partial}{\partial k}\varepsilon(k) + 2\nu k^2 E(k) \approx 0,$$

for wave numbers $l^{-1} << k << \eta^{-1}$, where l is the scale of the large eddies and η is the viscosity scale.

The Second Hypothesis
There exists a scaling (self-similarity)

- Thus we can define the energy dissipation rate in the equilibrium range

$$\varepsilon_0 = 2\nu \int_0^\infty k^2 E(k) dk$$

- The statistics are uniquely and universally determined by k and ε_0. From dimensional arguments, we get the Kolmogorov-Obukhov $(5/3)$ scaling law:

$$E(k) = C\varepsilon_0^{2/3} k^{-5/3}, \text{ for } l^{-1} << k << \eta^{-1},$$

where C is a constant.

- There also exist natural notions of (Kolmogorov's), length, velocity and time:

$$\eta = (\frac{\nu^3}{\varepsilon_0})^{1/4}; \;\; v = (\nu\varepsilon_0)^{1/4}; \;\; t = \frac{\eta}{v} = (\frac{\nu}{\varepsilon_0})^{1/2}$$

How do we solve the turbulence problem?

- We have to prove the two Hypothesis, that there exists a statistically stationary state for *Re* sufficiently large, and the Kolmogorov-Obukhov scaling.
- We have to include intermittency (events that happen occationally) in our solution.
- We must give a solution of the closure problem.
- Notice, this is not the millennium problem: To prove there exists a unique solution to the Navier-Stokes equation. The turbulence problem is a different problem.

Solution of the Stochastic Navier-Stokes

- We solve (1) using the Feynmann-Kac formula, and Girsanov's Theorem
- The solution is

$$
\begin{aligned}
u &= e^{Kt} e^{\int_0^t \nabla U ds} e^{\int_0^t dq} M_t u^0 \\
&+ \sum_{k \neq 0} \int_0^t e^{K(t-s)} e^{\int_0^{(t-s)} \nabla U dr} e^{\int_s^t dq} M_{t-s} \\
&\times (c_k^{1/2} db_s^k + d_k |k|^{1/3} ds) e_k(x)
\end{aligned}
$$

- K is the operator $K = \nu\Delta + \nabla\Delta^{-1} tr(\nabla u \nabla)$
- M_t is the Martingale

$$
M_t = e^{\{-\int_0^t (U+u)(B_s,s)\cdot dB_s - \frac{1}{2}\int_0^t |(U+u)(B_s,s)|^2 ds\}}
$$

- Using M_t as an integrating factor eliminates the inertial terms from the equation (1)

The Feynmann-Kac formula

- The Feynmann-Kac formula gives the exponential of a sum of terms of the form

$$\int_s^t dq^k = \int_0^t \int_{\mathbb{R}} \ln(1+h_k) N^k(dt,dz) - \int_0^t \int_{\mathbb{R}} h_k m^k(dt,dz),$$

  by a computation similar to the one that produces the geometric Lévy process [3, 4], m^k the Lévy measure.
- The form of the processes

$$e^{\int_0^t \int_{\mathbb{R}} \ln(1+h_k) N^k(dt,dz) - \int_0^t \int_{\mathbb{R}} h_k m^k(dt,dz)} = e^{N_t^k \ln \beta + \gamma \ln |k|} = |k|^\gamma \beta^{N_t^k}$$

  was found by She and Leveque [18], for $h_k = \beta - 1$
- It was pointed out by She and Waymire [19] and by Dubrulle [6] that they are log-Poisson processes.

Computation of the structure functions

The Kolmogorov-Obukhov-She-Leveque scaling

The scaling of the structure functions is

$$S_p \sim C_p |x-y|^{\zeta_p},$$

where

$$\zeta_p = \frac{p}{3} + \tau_p = \frac{p}{9} + 2(1-(2/3)^{p/3})$$

$\frac{p}{3}$ being the Kolmogorov scaling and τ_p the intermittency corrections. The scaling of the structure functions is consistent with Kolmogorov's $4/5$ law,

$$S_3 = -\frac{4}{5}\varepsilon |x-y|$$

to leading order, were $\varepsilon = \frac{d\mathcal{E}}{dt}$ is the energy dissipation

The first few structure functions

$$S_1(x,y,\infty) \leq \frac{2}{C} \sum_{k\in\mathbb{Z}^3\setminus\{0\}} \frac{|d_k|(1-e^{-\lambda_k t})}{|k|^{\zeta_1}} |\sin(\pi k\cdot(x-y))|$$

We get a stationary state as $t \to \infty$, and for $|x-y|$ small,

$$S_1(x,y,\infty) \sim \frac{2\pi^{\zeta_1}}{C} \sum_{k\in\mathbb{Z}^3\setminus\{0\}} |d_k||x-y|^{\zeta_1},$$

where $\zeta_1 = 1/3 + \tau_1 \approx 0.37$. Similarly,

$$S_2(x,y,\infty) \sim \frac{4\pi^{\zeta_2}}{C^2} \sum_{k\in\mathbb{Z}^3\setminus\{0\}} [{d_k}^2 + (\frac{C}{2})c_k]|x-y|^{\zeta_2},$$

when $|x-y|$ is small, where $\zeta_2 = 2/3 + \tau_2 \approx 0.696$.

Higher order structure functions

Similarly,

$$S_3(x,y,\infty) \sim \frac{2^3\pi}{C^3} \sum_{k\in\mathbb{Z}^3\setminus\{0\}} [|d_k|^3 + 3(C/2)c_k|d_k|]|x-y|.$$

For the pth structure functions, S_p is estimated [3, 4] by

$$S_p \le \frac{2^p}{C^p} \sum_{k\in\mathbb{Z}^3\setminus\{0\}} \frac{\sigma_k^p \cdot (-i\sqrt{2}\mathrm{sgn}\, M_k)^p\, U\left(-\frac{1}{2}p, \frac{1}{2}, -\frac{1}{2}(M_k/\sigma_k)^2\right)}{|k|^{\zeta_p}} \times |\sin^p(\pi k\cdot(x-y))|.$$

where U is the confluent hypergeometric function, $M_k = |d_k|(1-e^{-\lambda_k t})$ and $\sigma_k = \sqrt{(C/2)c_k(1-e^{-2\lambda_k t})}$.

KOSL Scaling of the Structure Functions, higher order $Re_\lambda \sim 16,000$

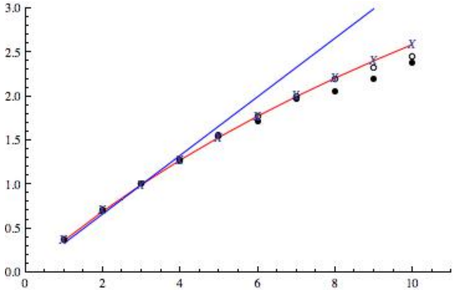


Figure: The exponents of the structure functions as a function of order, theory or Kolmogorov-Obukov-She-Leveque scaling (red), experiments (disks), dns simulations (circles), from [5], and experiments (X), from [18]. The Kolomogorov-Obukhov '41 scaling is also shown as a blue line for comparion.

KOSL Scaling of the Structure Functions, low order $Re_\lambda \sim 16,000$

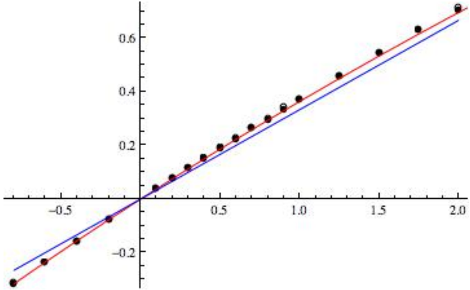


Figure: The exponents of the structure functions as a function of order $(-1, 2]$, theory or Kolmogorov-Obukov-She-Leveque scaling (red), experiments (disks), dns simulations (circles), from [5]. The Kolmogorov-Obukov '41 scaling is also shown as a blue line for comparion.

Outline

Computation of the Eddy Viscosity
(LES is similar)

- With the stochastic closure, we can now compute the eddy viscosity $\mathcal{R}_{ij} = \overline{u_i u_j}$, using the same method we used to compute the structure functions

$$
\begin{aligned}
\frac{\partial \overline{u u_j}}{\partial x_j} &= \frac{2}{C} e^{-\int_0^t (\nabla u + \nabla u^T) ds} \\
&\times \sum_{k>0} \frac{2\pi[(k \cdot c_k^{1/2}) c_k^{1/2} + (2/C)(k \cdot d_k) d_k]}{|k|^{\zeta_2}} e_k^2 \\
&\approx K |\nabla u|^{(1-\zeta_2)/2} e^{-\int_0^t (\nabla u + \nabla u^T) ds} \Delta^{(1-\zeta_2)/4} u
\end{aligned}
$$

$S = \frac{1}{2}(\nabla u + \nabla u^T)$ is the rate of strain tensor

- The first (multiplicative) term is an exponential (dynamic) Smagorinsky term

Outline

What object determines the whole statistical theory?

The invariant measure of turbulence

- Hopf [7] found a *functional differential equation* for the characteristic function of the invariant measure in 1952
- Kolmogorov computed the invariant measure in 1980s. It was a delta function supported at the origin
- Eberhard Hopf

The Invariant Measure and the Probability Density Functions (PDF)

- The Kolmogorov-Hopf equation for (1) is

$$\frac{\partial \phi}{\partial t} = \frac{1}{2}\mathrm{tr}[P_t C P_t^* \Delta \phi] + \mathrm{tr}[P_t \bar{D} \nabla \phi] + < K(z) P_t, \nabla \phi > \quad (3)$$

where $\bar{D} = (|k|^{1/3} D_k)$, $\phi(z)$ is a bounded function of z,

$$P_t = e^{-\int_0^t \nabla U \, dr} M_t \prod_k^m |k|^{2/3} (2/3)^{N_t^k}$$

- Variance and drift

$$Q_t = \int_0^t e^{K(s)} P_s C P_s^* e^{K^*(s)} ds, \quad E_t = \int_0^t e^{K(s)} P_s \bar{D} ds \quad (4)$$

The invariant measure of the stochastic Navier-Stokes

- The solution of the Kolmogorov-Hopf equation (3) is

$$R_t\phi(z) = \int_H \phi(e^{Kt}P_t z + E_t l + y)\mathcal{N}_{(0,Q_t)} * \mathbb{P}_{N_t}(dy)$$

- The invariant measure of the Navier-Stokes equation on $H_c = H^{3/2^+}(\mathbb{T}^3)$ is,

$$\begin{aligned}\mu(dx) &= e^{<Q^{-1/2}El,\ Q^{-1/2}x>-\frac{1}{2}|Q^{-1/2}El|^2}\mathcal{N}_{(0,Q)}(dx)\\ &\times \sum_k \delta_{k,l}\prod_{j\neq l}\delta_{N_t^j}\sum_{j=0}^{\infty} p^j_{m_l}\delta_{(N_l-j)}\end{aligned}$$

where $Q = Q_\infty$, $E = E_\infty$.

The differential equation for the PDF

- The quantity that can be compared directly to experiments is the PDF

$$E(\delta_j u) = E([u(x+s,\cdot) - u(x,\cdot)]\cdot r) = \int_{\infty}^{\infty} f_j(x)dx,$$

$j=1$, if $r=\hat{s}$ is the longitudinal direction, and $j=2$, $r=\hat{t}$, $t \perp s$ is a transversal direction

- We take the trace of the Kolmogorov-Hopf equation (3)
- The stationary equation satisfied by the PDF is

$$\frac{1}{2}\phi_{rr} + \frac{1+|c|r^d}{r}\phi_r = \phi, \tag{5}$$

where $d = \tau_p - \tau_{2p}$ is the intermittency index.

The Probability Density Function (PDF)

- The PDF, without intermittency corrections, is a Generalized Hyperbolic Distribution (GHD) of Barndorff-Nielsen [1]:

$$f(x) = \frac{(\gamma/\delta)^{\lambda+\frac{1}{2}}}{\sqrt{2\pi}K_{\lambda+\frac{1}{2}}(\delta\gamma)} \frac{K_\lambda\left(\alpha\sqrt{\delta^2+(x-\mu)^2}\right) e^{\beta(x-\mu)}}{\left(\sqrt{\delta^2+(x-\mu)^2}/\alpha\right)^{\lambda}} \tag{6}$$

- K_λ is modified Bessel's function of the second kind with index λ, $\gamma = \sqrt{\alpha^2-\beta^2}$. α, β, δ and μ are parameters.
- (6) is the solution of (5), with $d = 0$, and the PDF that can be compared a large class of experimental data.

O. Barndorff-Nielsen

Outline

The PDF of Turbulence

- The PDF becomes more complicated when the intermittency is included
- It becomes impossible to have a single PDF for all the different moments
- Instead one has to have a distribution that is a product of a discrete and continuous distributions
- The PDF of the velocity differences can be expressed as

$$\bar{\mu}(\cdot) = \int_{-\infty}^{\infty} \sum_{j=0}^{\infty} \frac{(\ln(|x|^{-6}))^j}{j!} |x|^6 \delta_{N_t^k - j}(\cdot) f(x) dx, \qquad (7)$$

where $f(x)$ is the Generalized Hyperbolic Distribution in Equation (6).

The pth Moment of the Velocity Differences

- The pth moment of the velocity difference is now easily computed:

$$\int_{-\infty}^{\infty}\sum_{j=0}^{\infty}\frac{(\ln(|x|^{-6}))^j}{j!}|x|^6\delta_{N_t^k-j}(|x|\left(\frac{2}{3}\right)^{N_t^k})^{\frac{p}{3}}f(x)dx$$
$$=\int_{-\infty}^{\infty}|x|^{3\xi_p}f(x)dx=\int_{-\infty}^{\infty}|x|^{p+3\tau_p}f(x)dx$$

- where
$$\xi_p=\frac{p}{3}+\tau_p$$
is the scaling exponent of the pth structure function, with the intermittency correction $\tau_p=2(1-(2/3)^{\frac{p}{3}})$
- Thus the pth moment is

$$\bar{\mu}(|\delta v|^p)=\int_{-\infty}^{\infty}|x|^{3\zeta_p}f(x)dx \tag{8}$$

Outline

Prandtl-von Kármán log-law for fluctuations

- The Prandtl-von Kármán log-law in the inertial range:

$$\langle u\rangle/u_\tau = \kappa^{-1}\ln(yu_\tau/\nu) + B, \tag{9}$$

- $u_\tau = \sqrt{\tau_w/\rho}$ is friction velocity, based on wall stress τ_w, κ the von Kármán constant and B a constant
- Marusic and Kunkel [13], Hultmark et al. [8] and Marusic [12], Marusic et al. [15] and Meneveau and Marusic [14] proposed a universal log-law, for the streamwise fluctuations $u' = (u - \langle u\rangle)/u_\tau$:

$$\langle (u')^{2p}\rangle^{1/p} = B_p - A_p\ln(y/\delta) = D_p(Re_\tau) - A_p\ln(y^+) \tag{10}$$

motivated by the "attached eddy hypothesis" of Townsend [20]

The coefficients of the log-law for the fluctuations

- The stochastic closure shows that the universal constants satisfy the relationship

$$A_p = \frac{C_p^{1/p}}{C_1} \left(\frac{1}{l^*}\right)^{\zeta_1 - \frac{\zeta_p}{p}} A_1 .$$

where $\zeta_p = p/3 + \tau_p = p/9 + 2(1-(2/3)^{p/3})$ are the (KOSL) scaling exponents of the structure functions and l^* is a small number

- The sub-Gaussian behavior of the A_ps is caused by the KOSL scaling
- Moreover, if A_1 is finite, then all the $A_p s$ are bounded:

$$A_p \to \left(\frac{1}{l^*}\right)^{\zeta_1 - 1/9} \lim_{p\to\infty} \frac{C_p^{1/p}}{C_1} A_1 = b\, A_1$$

as $p \to \infty$, where b is a constant.

Plot of A_p as a function of p

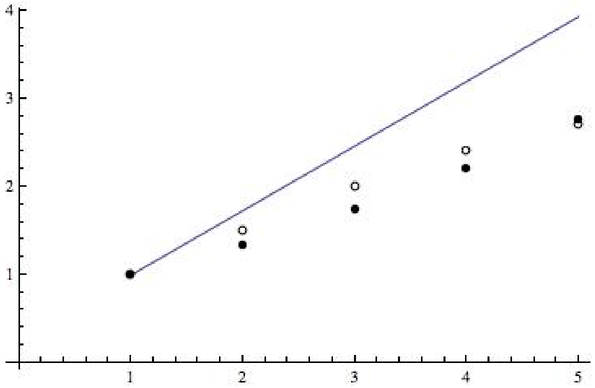


Figure: The first few coefficients A_p as functions of $2p$ (black dots), compared with data (white dots) with Reynolds number $Re_\tau = 19,030$. The blue line represents the Gaussian case.

Plot of A_p as a function of p, for large p

However, the figure shows, that the decay of A_p to bA_1 only takes place for very large p.

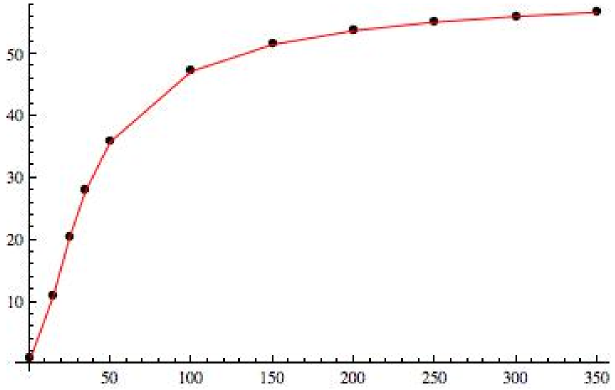
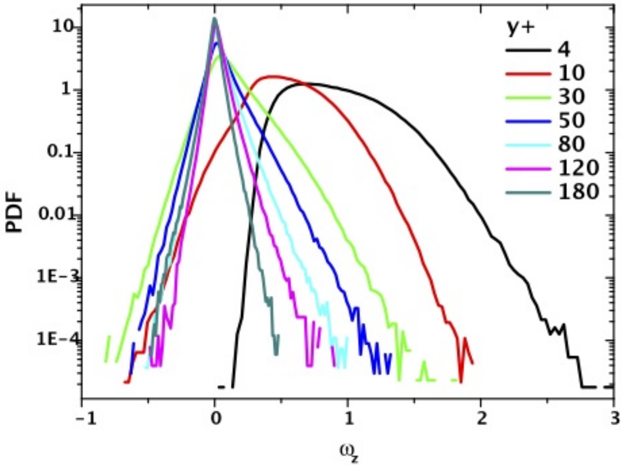


Figure: The coefficients A_p for large values of p.

Log of PDFs for the spanwise vorticity
GHDs in the inertial range, Gaussian in the viscous range



Outline

Conclusions

- The classical Reynolds decomposition (RANS) of turbulent flow can be closed by a stochastic Navier-Stokes equation for the small scale flow
- The form of the noise in the small scale flow is determined by the central limit theorem, the large deviations principle (additive) and a simple jump process multiplying the velocity (multiplicative)
- The multiplicative noise gives, by the Feynmann-Kac formula, the log-Poisson processes of She-Leveque, Waymire and Dubrulle, that produce the intermittency
- The estimate of the structure functions gives the Kolmogorov-Obukhov-She-Leveque scaling, including the intermittency corrections
- The eddy viscosity can be computed from the small scale flow, solving the closure problem

More Conclusions

- The statistics of the eddy viscosity can now be compared to the statistics of subgrid models for LES
- The Kolmogorov-Hopf equation of the stochastic Navier-Stokes equation is found and its solution gives the invariant measure of turbulence
- The measure gives the Generalized Hyperbolic distributions as PDFs when projected, however, different moments have different parameters, because of intermittency
- The PDF of Turbulence is a product of a discrete (intermittency) and continuous (K-O '41) probability distribution
- These methods extend to boundary flows, and permit a computations of the coefficients in the generalized Prandtl-von Kármán law for the velocity fluctuation
- Lagrangian turbulence is still wide open

Lykke til ! Ole

Comparison with Simulations and Experiments

- We now compare the above PDFs with the PDFs found in simulations and experiments.
- The direct Navier-Stokes (DNS) simulations were provided by Michael Wilczek from his Ph.D. thesis, see [21].
- The experimental results are from Eberhard Bodenschatz and Xu Haitao in Göttingen.
- We thank both for the permission to use these results to compare with the theoretically computed PDFs.
- A special case of the hyperbolic distribution, the NIG distribution, was used by Barndorff-Nielsen, Blaesild and Schmiegel [2] to obtain fits to the PDFs for three different experimental data sets.

The PDF from simulations and fits for the longitudinal direction

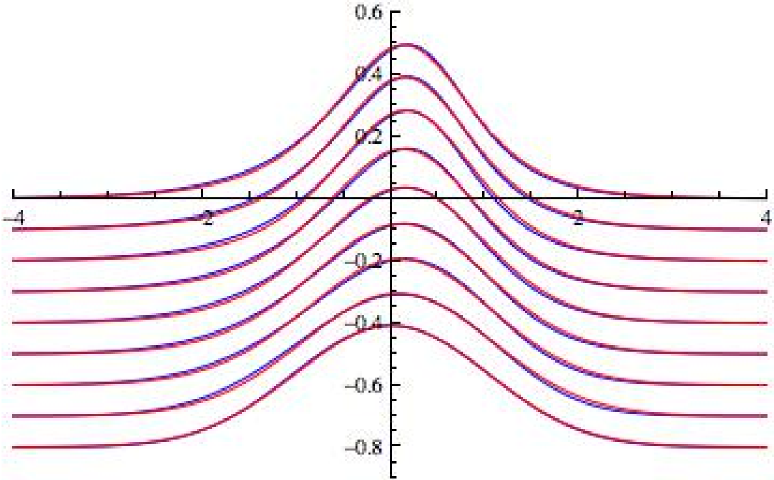


Figure: The PDF from simulations and fits for the longitudinal direction.

The log of the PDF from simulations and fits for the longitudinal direction

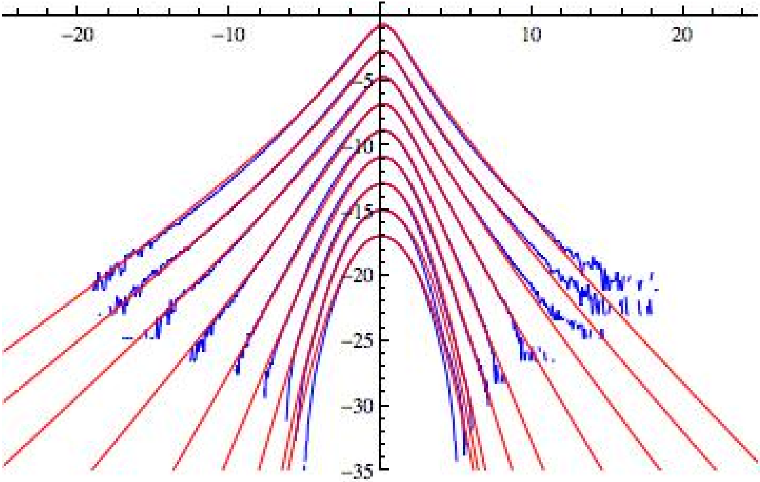


Figure: The log of the PDF from simulations and fits for the longitudinal direction, compare Fig. 4.5 in [21].

The PDF from simulations and fits for the transversal direction

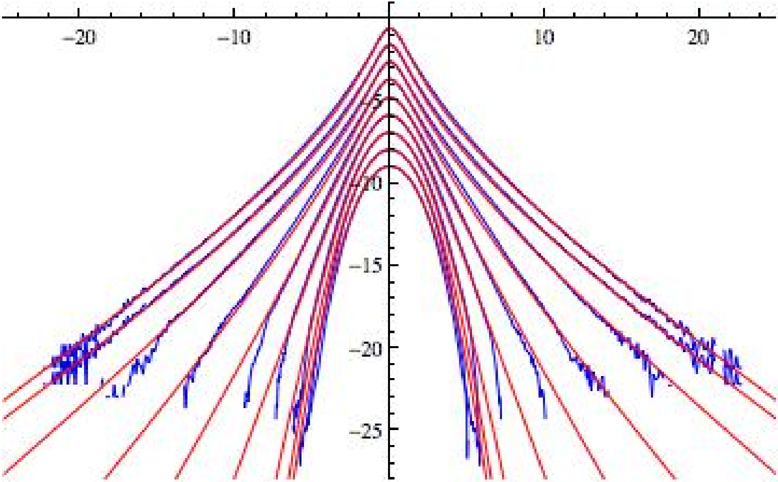


Figure: The log of the PDF from simulations and fits for the a

The PDFs from experiments and fits

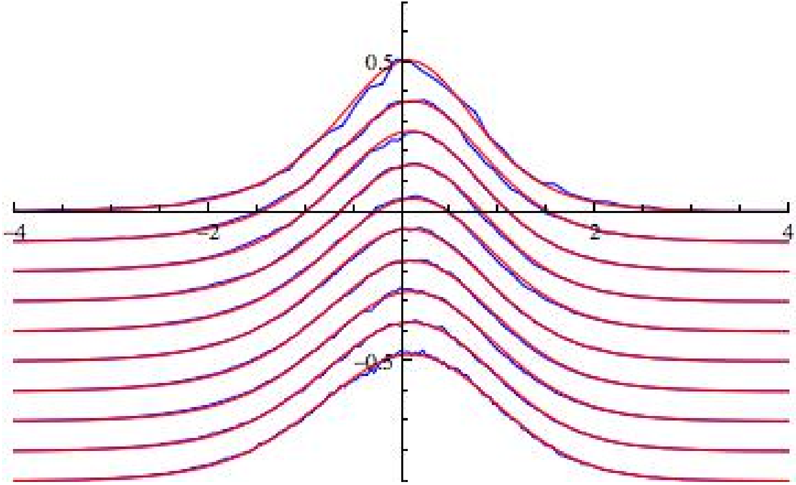


Figure: The PDFs from experiments and fits.

The log of the PDFs from experiments and fits

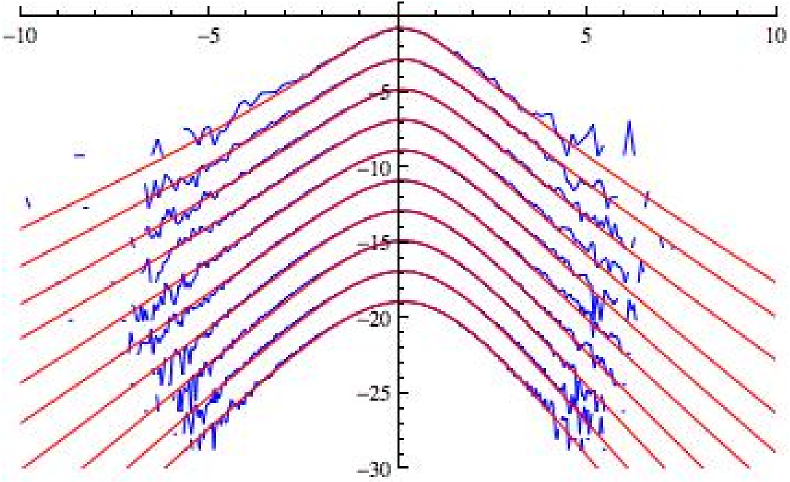


Figure: The log of the PDFs from experiments and fits.

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