Representation of ambit fields

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Århus, June 19, 2015

Introduction

- Study ambit fields as Volterra processes in Hilbert space
- Consider representations of ambit fields
  - Series representations as LSS processes
  - Solutions of SPDEs in Hilbert space

Hilbert-valued ambit fields

–Volterra processes in Hilbert space–

Definition of "classical" ambit fields

$$X(t,x)=\int_{-\infty}^{t}\int_{\mathcal{A}} g(t-s,x,y)\sigma(s,y)\,L(ds,dy)$$

- L is a *Lévy basis*
- g non-negative deterministic function, $g(u,x,y)=0$ for $u<0$.
- Stochastic volatility process σ independent of L, stationary
- $\mathcal{A}$ a Borel subset of $\mathbb{R}^d$: "ambit" set

- L is a *Lévy basis* on $\mathbb{R}^d$ if
  1. the law of $L(A)$ is infinitely divisible for all bounded sets A
  2. if $A \cap B = \emptyset$, then $L(A)$ and $L(B)$ are independent
  3. if $A_1, A_2, \ldots$ are disjoint bounded sets, then

$$L(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} L(A_i)\, , a.s$$

- We restrict to zero-mean, and square integrable Lévy bases L
- Use Walsh's definition of the stochastic integral

- Application of ambit fields:
  - Turbulence
  - Tumor growth
  - Finance: fixed-income and energy
- Our goals: lift the ambit fields to processes in Hilbert space
  - Hilbert-space valued ambit fields
- ..and to analyse representations of such!

- Define $\mathcal{H}$-valued process $t \mapsto X(t)$

$$X(t) = \int_0^t \Gamma(t,s)(\sigma(s))\,dL(s)$$

- $\mathcal{U}, \mathcal{V}, \mathcal{H}$ three separable Hilbert spaces
- $s \mapsto L(s)$ $\mathcal{V}$-valued Lévy process
  - Square integrable, covariance operator $\mathcal{Q} \in \mathcal{L}_{\mathrm{HS}}(\mathcal{V})$
  - Mean zero (L is $\mathcal{V}$-martingale)
- $s \mapsto \sigma(s)$ predictable process with values in $\mathcal{U}$
  - Stochastic volatility or intermittency
- $(t,s) \mapsto \Gamma(t,s)$, $s \leq t$, $\mathcal{L}(\mathcal{U}, \mathcal{L}(\mathcal{V}, \mathcal{H}))$-valued measurable mapping
  - Non-random *kernel* function

- Integrability condition for Γ and σ:

$$\mathbb{E}\left[\int_0^t \|\Gamma(t,s)(\sigma(s))\mathcal{Q}^{1/2}\|_{\mathrm{HS}}^2 \, ds\right] < \infty$$

- We call X a *$\mathcal{H}$ambit field*

- A sufficient integrability condition:

Lemma: If

$$\int_0^t \|\Gamma(t,s)\|_{\mathrm{op}}^2 \mathbb{E}\left[|\sigma(s)|_{\mathcal{U}}^2\right] \, ds < \infty$$

where $\|\cdot\|_{\mathrm{op}}$ operator norm on $\mathcal{L}(\mathcal{U}, \mathcal{L}(\mathcal{V}, \mathcal{H}))$, the integrability condition for the $\mathcal{H}$ambit field holds.

"Proof": Use linearity of $\Gamma(t,s)$:

$$\|\Gamma(t,s)(\sigma(s))\mathcal{Q}^{1/2}\|_{\mathrm{HS}}^2 = \sum_{m=1}^{\infty} |\Gamma(t,s)(\sigma(s))\mathcal{Q}^{1/2} v_m|_{\mathcal{V}}^2$$

Characteristic functional

Proposition: Suppose that σ is independent of L. For $h \in \mathcal{H}$ it holds

$$\mathbb{E}\left[\exp(\mathrm{i}(h, X(t))_{\mathcal{H}})\right] = \mathbb{E}\left[\exp\left(\int_0^t \Psi_L((\Gamma(t,s)(\sigma(s)))^* h)\right)\, ds\right]$$

where Ψ_L is the characteristic exponent of $L(1)$.

"Proof": Condition on σ, and use the independent increment property of L along with the fact

$$(h, \Gamma(t,s)(\sigma(s))\Delta L(s))_{\mathcal{H}} = ((\Gamma(t,s)(\sigma(s)))^* h, \Delta L(s))_{\mathcal{V}}$$

- Example: $L = W$, $\mathcal{V}$-valued Wiener process
- For $v \in \mathcal{V}$,

$$\Psi_W(v) = -\frac{1}{2}(\mathcal{Q}v, v)_{\mathcal{V}}$$

- Characteristic function of X (Bochner ds-integral)

$$\mathbb{E}\left[\exp(\mathrm{i}(h, X(t))_{\mathcal{H}})\right]$$
$$= \mathbb{E}\left[\exp\left(-\frac{1}{2}(h, \int_0^t \Gamma(t,s)(\sigma(s))\mathcal{Q}(\Gamma(t,s)(\sigma(s)))^* h\, ds)_{\mathcal{H}}\right)\right]$$

- X is conditional Gaussian

Examples

Example: from $\mathcal{H}$ambit to ambit

- Let $\mathcal{A} \subset \mathbb{R}^n$ Borel set, $\mathcal{U}$ a Hilbert space of real-valued functions on $\mathcal{A}$
- Let $(t, s, x, y) \mapsto g(t, s, x, y)$ be a measurable real-valued function for $0 \leq s \leq t \leq T$, $y \in \mathcal{A}$, $x \in \mathcal{B}$, $\mathcal{B} \subset \mathbb{R}^d$
- Suppose $\mathcal{V}$ is a Hilbert space of absolutely continuous functions on $\mathcal{A}$.
- Define for $\sigma \in \mathcal{U}$ the linear operator on $\mathcal{V}$

$$\Gamma(t, s)(\sigma) := \int_{\mathcal{A}} g(t, s, \cdot, y)\sigma(y)$$

acting on $f \in \mathcal{V}$ as

$$\Gamma(t, s)(\sigma)f = \int_{\mathcal{A}} g(t, s, \cdot, y)\sigma(s, y)f(dy) \, .$$

- Let $\mathcal{H}$ be a Hilbert space of real-valued functions on $\mathcal{B}$
- Let L be a $\mathcal{V}$-valued Lévy process, σ $\mathcal{U}$-valued predictable process
  - Suppose integrability conditions on $s \mapsto \Gamma(t,s)(\sigma(s))$
- $X(t,x)$ is an ambit field

$$X(t,x) = \int_0^t \int_{\mathcal{A}} g(t,s,x,y)\sigma(y)\, L(ds,dy)$$

- Example of Hilbert space?

Realization in Filipovic space

- Let $\mathcal{U} = \mathcal{V} = \mathcal{H}$, $n = d = 1$, $\mathcal{A} = \mathcal{B} = \mathbb{R}_+$
- Let $w \in C^1(\mathbb{R}_+)$ be non-decreasing, $w(0) = 1$ and $w^{-1} \in L^1(\mathbb{R}_0+)$
- Let $\mathcal{U} := H_w$ be the space of absolutely continuous functions on $\mathbb{R}_+$ where

$$|f|_w^2 = f^2(0) + \int_{\mathbb{R}_+} w(y)|f'(y)|^2 \, dy < \infty$$

- H_w separable Hilbert space.
  - Introduced by Filipovic (2001)
  - Main application: realization of forward rate HJM models

Hilbert-valued OU with stochastic volatility

- Fix $\mathcal{V} = \mathcal{H}$, and let $\mathcal{A}$ unbounded operator on $\mathcal{H}$ with C_0-semigroup $\mathcal{S}_t$.
- B $\mathcal{H}$-valued Wiener process with covariance operator $\mathcal{Q}$.
- B. Rüdiger and Süss (2015): Let $\sigma(t)$ be a $\mathcal{U} := L_{HS}(\mathcal{H})$-valued predictable process,

$$dX(t) = \mathcal{A}X(t)\,dt + \sigma(t)\,dB(t)$$

- Mild solution

$$X(t) = \mathcal{S}_t X(0) + \int_0^t \mathcal{S}_{t-s}\sigma(s)\,dB(s)$$

- X as $\mathcal{H}$ambit field: define $\Gamma(t,s) \in \mathcal{L}(L_{HS}(\mathcal{H}), \mathcal{L}(\mathcal{H}))$

$$\Gamma(t,s) : \sigma \mapsto \mathcal{S}_{t-s}\sigma$$

- A BNS SV model: $\sigma(t) = \mathcal{Y}^{1/2}(t)$

$$d\mathcal{Y}(t) = \mathbb{C}\mathcal{Y}(t)\,dt + d\mathcal{L}(t)$$

- $\mathbb{C} \in \mathcal{L}(L_{HS}(\mathcal{H}))$, with C_0-semigroup $\mathbb{S}_t$
- $\mathcal{L}$ is a $L_{HS}(\mathcal{H})$-valued "subordinator"

- $\mathcal{Y}(t)$ symmetric, positive definite, $L_{HS}(\mathcal{H})$-valued process,

$$\mathbb{E}[\|\sigma(t)\|_{\mathcal{U}}^2] = \sum_{n=1}^{\infty}(\sigma(t)h_k, \sigma(t)h_k)_{\mathcal{H}} = \sum_{k=1}^{\infty}(\sigma^2(t)h_k, h_k)_{\mathcal{H}} = \mathrm{Tr}(\mathcal{Y}(t))$$

- The trace is continuous, and integrability holds for well-definedness of X

$$\mathrm{Tr}(\mathcal{Y}(t)) = \mathrm{Tr}(\mathbb{S}_t\mathcal{Y}_0) + \mathrm{Tr}(\int_0^t \mathbb{S}_s\, ds\mathbb{E}[\mathcal{L}(1)])$$

- Infinite-dimensional extension of Ole and Stelzer (2007)

$\mathcal{H}$ambit fields as Lévy semistationary (LSS) processes

- Let $\{u_n\}, \{v_m\}$ and $\{h_k\}$ be ONB in $\mathcal{U}, \mathcal{V}$ and $\mathcal{H}$ resp.
  - Recall separability of the Hilbert spaces
- $L_m := (L, v_m)_{\mathcal{V}}$ are $\mathbb{R}$-valued Lévy processes
  - zero mean, square integrable
  - but, not independent nor zero correlated
- Define LSS processes $Y_{n,m,k}(t)$ by

$$Y_{n,m,k}(t) = \int_0^t g_{m,n,k}(t,s)\sigma_n(s)\,dL_m(s)$$

$$g_{n,m,k}(t,s) := (\Gamma(t,s)(u_n)v_m, h_k)_{\mathcal{H}} \qquad \sigma_n(s) := (\sigma(s), u_n)_{\mathcal{U}}$$

Proposition: Assume

$$\int_0^t \|\Gamma(t,s)\|_{\mathrm{op}}^2 \left(\sum_{n=1}^{\infty} \mathbb{E}[\sigma_n^2(s)]^{1/2} \right)^2 ds < \infty$$

then,

$$X(t) = \sum_{n,m,k=1}^{\infty} Y_{n,m,k}(t) h_k$$

"Proof": Expand all elements along the ONB's in their respective spaces. The integrability assumption ensures the commutation of an infinite sum and stochastic integral wrt. L_m (A stochastic Fubini theorem).

- Ole et al. (2013): energy spot price modeling using LSS processes
    - Finite factors
    - Implied forward prices become scaled finite sums of LSS processes
- Ole et al. (2014): energy forward prices as ambit fields
    - Infinite LSS factor models!
- B. Krühner (2014): HJM forward price dynamics representable as countable scaled sums of OU process
    - Possibly complex valued OU processes

- Integrability condition implies the sufficient condition for existence of $\mathcal{H}$ambit field:
- By Parseval's identity

$$\mathbb{E}[\|\sigma(s)\|_{\mathcal{U}}^2] = \sum_{n=1}^{\infty} \mathbb{E}[(\sigma(s), u_n)_{\mathcal{U}}^2]$$

- Sufficient condition for LSS representation: there exists $a_n > 0$ s.t. $\sum_{n=1}^{\infty} a_n^{-1} < \infty$ and

$$\sum_{n=1}^{\infty} a_n \int_0^t \|\Gamma(t,s)\|_{\text{op}}^2 \mathbb{E}[(\sigma(s), u_n)_{\mathcal{U}}^2] \, ds < \infty$$

$\mathcal{H}$ambit fields and SPDEs

- Known connection between an LSS process and the boundary of a hyperbolic stochastic partial differential equation (SPDE):

$$dZ(t,x) = \partial_x Z(t,x)\,dt + g(t+x,t)\sigma(t)\,dL(t)$$

$$Z_0(t) := Z(t,0) = \int_0^t g(t,s)\sigma(s)\,dL(s)$$

- L $\mathbb{R}$-valued Lévy process, $x \geq 0$
- Goal here to show similar result for $\mathcal{H}$ambit fields!
- B. Eyjolfsson (2015+) deviced iterative (finite difference) numerical schemes using this relationship

- Assume $\widetilde{\mathcal{H}}$ a Hilbert space of strongly measurable $\mathcal{H}$-valued functions on $\mathbb{R}_+$
- Suppose $\mathcal{S}_\xi$ right-shift operator is C_0-semigroup on $\widetilde{\mathcal{H}}$

$$\mathcal{S}_\xi f := f(\xi + \cdot)\,, \quad f \in \widetilde{\mathcal{H}}$$

- Generator is $\partial_\xi = \partial / \partial \xi$
- Consider hyperbolic SPDE in $\widetilde{\mathcal{H}}$

$$\mathcal{X}(t) = \partial_\xi \mathcal{X}(t)\, dt + \Gamma(t + \cdot, t)(\sigma(t))\, dL(t)\,, \mathcal{X}(0) \in \widetilde{\mathcal{H}}$$

- Predictable $\widetilde{\mathcal{H}}$-valued unique solution

$$\mathcal{X}(t) = \mathcal{S}_t \mathcal{X}(0) + \int_0^t \mathcal{S}_{t-s} \Gamma(s + \cdot, s)(\sigma(s))\, dL(s)$$

Proposition: Assume that the evaluation map $\delta_x : \widetilde{\mathcal{H}} \to \mathcal{H}$ defined by $\delta_x f = f(x) \in \mathcal{H}$ for every $x \geq 0$ and $f \in \widetilde{\mathcal{H}}$ is a continuous linear operator. If $\mathcal{X}(0) = 0$, Then $X(t) = \delta_0(\mathcal{X}(t))$.

"Proof": Argue that

$$\delta_0 \int_0^t \Gamma(t + \cdot, s)(\sigma(s))\, dL(s) = \int_0^t \Gamma(t, s)(\sigma(s))\, dL(s)$$

- Need a space $\widetilde{\mathcal{H}}$ with $\delta_x \in \mathcal{L}(\widetilde{\mathcal{H}}, \mathcal{H})$

Abstract Filipovic space

- $f \in L^1_{loc}(\mathbb{R}_+, \mathcal{H})$ is *weakly differentiable* if there exists $f' \in L^1_{loc}(\mathbb{R}_+, \mathcal{H})$ such that

$$\int_{\mathbb{R}_+} f(x)\phi'(x)\,dx = -\int_{\mathbb{R}_+} f'(x)\phi(x)\,dx\,, \forall \phi \in C_c^\infty(\mathbb{R}_+)$$

- Integrals interpreted in Bochner sense
- Let $w \in C^1(\mathbb{R}_+)$ be a non-decreasing function with $w(0) = 1$ and

$$\int_{\mathbb{R}_+} w^{-1}(x)\,dx < \infty$$

- Define $\mathcal{H}_w$ to be the space of $f \in L^1_{loc}(\mathbb{R}_+, \mathcal{H})$ for which there exists $f' \in L^1_{loc}(\mathbb{R}_+, \mathcal{H})$ such that

$$\|f\|_w^2 = |f(0)|_{\mathcal{H}}^2 + \int_{\mathbb{R}_+} w(x)|f'(x)|_{\mathcal{H}}^2 \, dx < \infty \, .$$

- $\mathcal{H}_w$ is a separable Hilbert space with inner product

$$\langle f, g \rangle_w = (f(0), g(0))_{\mathcal{H}} + \int_{\mathbb{R}_+} w(x)(f'(x), g'(x))_{\mathcal{H}} \, dx$$

- Fundamental theorem of calculus: If $f \in \mathcal{H}_w$, then $f' \in L^1(\mathbb{R}_+, \mathcal{H})$, $\|f'\|_1 \leq c\|f\|_w$, and

$$f(x+t) - f(x) = \int_x^{x+t} f'(y)\,dy$$

- Shift-operator $\mathcal{S}_\xi, \xi \geq 0$ is *uniformly bounded*

$$\|\mathcal{S}_\xi f\|_w^2 \leq 2(1+c^2)\|f\|_w^2$$

- Constant equal to $c^2 = \int_{\mathbb{R}_+} w^{-1}(x)\,dx$

Lemma: Evaluation map $\delta_x : \mathcal{H}_w \to \mathcal{H}$ is a linear bounded operator with

$$|\delta_x f|_{\mathcal{H}} \leq K\|f\|_w$$

"Proof": FTC, Bochner's norm inequality and Cauchy-Schwartz inequality yield

$$|\delta_x f|^2_{\mathcal{H}} = |f(x)|^2_{\mathcal{H}} \leq 2|f(0)|^2_{\mathcal{H}} + 2\int_{\mathbb{R}_+} w^{-1}(y)\,dy \int_{\mathbb{R}_+} w(y)|f'(y)|^2_{\mathcal{H}}\,dy$$

- We have an example $\widetilde{\mathcal{H}} = \mathcal{H}_w$!

Classical and abstract Filipovic space

Proposition: For $\mathcal{L} \in \mathcal{H}^*$, $x \mapsto \mathcal{L} \circ \delta_x(g) = \mathcal{L}(g(x)) \in H_w$ for $g \in \mathcal{H}_w$. Moreover, if $h_x(y) = 1 + \int_0^{x \wedge y} w^{-1}(z)\, dz$ and $\ell_x = \mathcal{L}^*(h_x)$, then

$$\mathcal{L}(g(x)) = \langle g, \ell_x \rangle_w$$

"Proof": Follows from linearity of $\mathcal{L}$, FTC and Bochner's norm inequality. Further, if $\bar{\delta}_x$ is the evaluation map on H_w, then $\bar{\delta}_x(v) = (v, h_x)_w$, $v \in H_w$.

Til lykke med din 80 års fødselsdag, Ole!

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