Limit theorems for stationary increments Lévy driven moving averages

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Power variations

- For a stochastic process $X = (X_t)_{t \geq 0}$ and $k \geq 1$ we define the *k-th order increments* of X via

$$\Delta_{i,k}^n X := \sum_{j=0}^{k} (-1)^j \binom{k}{j} X_{(i-j)/n}.$$

For instance,

$$\Delta_{i,1}^n X = X_{i/n} - X_{(i-1)/n} \quad \text{and} \quad \Delta_{i,2}^n X = X_{i/n} - 2X_{(i-1)/n} + X_{(i-2)/n}.$$

- The power variation of k-th order increments of X is given by the statistic

$$V(X, p, k)_n := \sum_{i=k}^{n} |\Delta_{i,k}^n X|^p.$$

In the following we will study the asymptotic behaviour of the functional $V(X, p, k)_n$ as $n \to \infty$.

Power variation for the fractional Brownian motion: First order asymptotics

Let B^H be a fractional Brownian motion with Hurst exponent $H \in (0,1)$, that is, a centered Gaussian process with covariance function

$$\mathrm{Cov}(B_t^H, B_s^H) = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}).$$

By the ergodic theorem it follows that :
First order asymptotics for B^H:

$$n^{-1+pH} V(B^H, p, k)_n \xrightarrow{\mathbb{P}} m_{p,k} := \mathbb{E}[|\Delta_{1,k}^1 B^H|^p].$$

Second order asymptotics

Theorem (Breuer–Major [1], Taqqu [2])

The following assertions hold:

(i) *Assume that* $k = 1$ *and* $H \in (0, 3/4)$, *or* $k \geq 2$. *Then*

$$\sqrt{n}\left(n^{-1+pH} V(B^H, p, k)_n - m_{p,k}\right) \xrightarrow{d} \mathcal{N}(0, v_{p,k}).$$

(ii) *When* $k = 1$ *and* $H \in (3/4, 1)$ *it holds that*

$$n^{2-2H}\left(n^{-1+pH} V(B^H, p, k)_n - m_{p,k}\right) \xrightarrow{d} Z,$$

where Z *is a Rosenblatt random variable.*

[1] Breuer and Major (1983). Central limit theorems for nonlinear functionals of Gaussian fields. *Journal of Multivariate Analysis* 13.
[2] Taqqu (1979). Convergence of integrated processes of arbitrary Hermite rank. *Z. Wahrsch. Verw. Gebiete* 50.

Extension

In a series of papers [1], [2] and [3] the above mentioned results have been extended to more general processes $(X_t)_{t\geq 0}$ of the form

$$X_t = \int_{-\infty}^{t} g(t-s)\,\sigma_s\,dB_s \qquad \text{or} \qquad X_t = \int_0^t \sigma_s\,dG_s$$

where B is a Brownian motion, σ predictable stochastic process, $g : \mathbb{R} \to \mathbb{R}$ deterministic function and G is a stationary increment Gaussian process.

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[1] O.E. Barndorff-Nielsen, J.M. Corcuera and M. Podolskij (2009): Power variation for Gaussian processes with stationary increments. Stochastic Processes and Their Applications 119, 1845–865.

[2] O.E. Barndorff-Nielsen, J.M. Corcuera and M. Podolskij (2011): Multipower variation for Brownian semistationary processes. Bernoulli 17(4), 1159–1194.

[3] O.E. Barndorff-Nielsen, J.M. Corcuera and M. Podolskij (2013): Limit theorems for functionals of higher order differences of Brownian semi-stationary processes. In Prokhorov and Contemporary Probability Theory.

Very little is known outside the two settings:

1. Itô semimartingales
2. Gaussian driven processes.

Two exceptions are the two works

1. The work [1] on the quadratic variation of the Rosenblatt process.
2. The work [2] on power variation of a class of fractional Lévy processes.

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[1] C. Tudor and F. Viens (2009). Variations and estimators for self-similarity parameters via Malliavin calculus. *Ann. Probab.* 37.

[2] A. Benassi, S. Cohen and J. Istas (2004). On roughness indices for fractional fields. *Bernoulli* 10(2), 357–373.

Background on Lévy processes

- Lévy processes are stochastic processes with *stationary* and *independent* increments, which are *continuous in probability*.
- A symmetric Lévy process L, with no Gaussian component and Lévy measure ν, has the characteristic function

$$\mathbb{E}[e^{iuL_t}] = e^{t\psi(u)}$$

with

$$\psi(u) = \int_{\mathbb{R}} (e^{iux} - 1 - iux1_{\{|x|\leq 1\}})\,\nu(dx).$$

Blumenthal-Getoor index

- Let $\Delta L_s = L_s - L_{s-}$ denote the jump of L at time s. The Blumenthal-Getoor index β is defined as

$$\beta := \inf \left\{ r \geq 0 : \int_{-1}^{1} |x|^r \nu(dx) < \infty \right\}.$$

- A symmetric β-stable Lévy process (SβS) with $\beta \in (0, 2)$, has a Lévy measure of the form

$$\nu(dx) = k_0 |x|^{-1-\beta} dx.$$

- For β-stable Lévy processes it holds that

$$\beta = \text{Blumenthal-Getoor index.}$$

Model: Lévy moving averages

- We consider a *stationary increment Lévy driven moving average* (SIMA)

$$X_t = \int_{-\infty}^{t} \{g(t-s) - g_0(-s)\} dL_s,$$

  where L is a Lévy process without a Gaussian component, and g is a deterministic function.
- Process X is an infinitely divisible process with stationary increments.
- For $g_0 = 0$, X is a moving average and is stationary.

Fractional Lévy processes

- In the special case $g(t) = g_0(t) = t_+^\alpha$, X is called a *fractional Lévy process* and has the form

$$X_t = \int_{-\infty}^{t} \{(t-s)^\alpha - (-s)_+^\alpha\}\, dL_s.$$

- If in addition, L is an β-stable Lévy process then X is the *linear fractional stable motion* with Hurst index $H = \alpha + 1/\beta$. Here X is self-similar with index H, i.e. for all $a > 0$

$$(X_{at})_{t \geq 0} \stackrel{\mathcal{D}}{=} (a^H X_t)_{t \geq 0}.$$

  For $\beta = 2$, X is the fractional Brownian motion.

The semimartingale property of X

Consider a fractional Lévy process X:

$$X_t = \int_{-\infty}^{t} \{(t-s)^{\alpha} - (-s)_+^{\alpha}\}\, dL_s.$$

It follows from [1] that X is a semimartingale if and only if

$$\int_{|x|\le 1} |x|^{\frac{1}{1-\alpha}}\, \nu(dx) < \infty.$$

In particular,

$$\begin{aligned} \alpha > 1 - 1/\beta \quad &\Rightarrow \quad X \text{ is a semimartingale} \\ \alpha < 1 - 1/\beta \quad &\Rightarrow \quad X \text{ is not a semimartingale.} \end{aligned}$$

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[1] B. and J. Rosiński (2014). On infinitely divisible semimartingales. *Probab. Theory Relat. Fields*.

Assumptions on $X_t = \int_{-\infty}^t \{g(t-s) - g_0(-s)\}\, dL_s$

Assumption (A):

(i): $g(x) \sim c_0 x^\alpha$ as $x \to 0$, with $\alpha > 0$ and $c_0 \neq 0$.

(ii): For some $\theta \in (0,2)$ it holds that

$$\limsup_{t\to\infty} t^\theta \nu((-t,t)^c) < \infty.$$

(iii): $g \in C^k((0,\infty))$, and for some $\delta > 0$,

$$|g^{(k)}(x)| \leq K|x|^{\alpha-k}, \qquad x \in (0,\delta),$$

$g^{(k)} \in L^\theta((\delta,\infty))$, and $|g^{(k)}|$ is decreasing on (δ,∞). And $g - g_0 \in L^\theta(\mathbb{R})$.

Assumption (A) guarantees the existence of process X.

Remarks

- We will see that the limit theory for power variation

$$V(p,k)_n = \sum_{i=k}^{n} |\Delta_{i,k}^n X|^p \qquad \text{as } n \to \infty$$

gives quite surprising results. In particular, it depends on the interplay between the four parameters

$$\underbrace{k}_{\text{order of increments}} \quad \underbrace{p}_{\text{power}} \quad \underbrace{\alpha}_{\text{behaviour of } g \text{ at } 0} \quad \text{and} \quad \underbrace{\beta}_{\text{BG-index of } L}$$

First order asymptotics for $V(p,k)_n = \sum_{i=k}^n |\Delta_{i,k}^n X|^p$

Theorem (B., Lachièze-Rey and Podolskij)

Assume that assumption (A) holds and L is a Lévy process without Gaussian component with Blumenthal-Getoor index $\beta \in (0,2)$.

(i): If $\alpha \in (0, k - 1/p)$ *and* $p > \beta$, *we obtain as* $n \to \infty$

$$n^{\alpha p} V(p,k)_n \xrightarrow{\mathcal{L}-s} |c_0|^p \sum_{m:\ T_m \in [0,1]} |\Delta L_{T_m}|^p V_m$$

where $(T_m)_{m \geq 1}$ *are jump times of L,* $(U_m)_{m \geq 1}$ *is an i.i.d. sequence of* $\mathcal{U}([0,1])$*-distributed random variables independent of L,*

$$h_k(x) := \sum_{j=0}^{k} (-1)^j \binom{k}{j} (x - j)_+^{\alpha},$$

$$V_m := \sum_{l=0}^{\infty} |h_k(l + U_m)|^p.$$

Theorem

(i): If $\alpha \in (0, k - 1/p)$ and $p > \beta$, then

$$n^{\alpha p} V(p,k)_n \xrightarrow{\mathcal{L}-s} |c_0|^p \sum_{m:\, T_m \in [0,1]} |\Delta L_{T_m}|^p V_m := Z$$

1. The limit Z is infinitely divisible with Lévy measure

$$(\nu \otimes \eta) \circ \big((y,v) \mapsto |c_0 y|^p v\big)^{-1}$$

   where η denotes the law of

$$V_1 = \sum_{l=0}^{\infty} |h_k(l + U_1)|^p.$$

2. Convergence in probability does not hold.

First order asymptotics for power variation

Theorem (cont.)

(ii): Assume that L is a $S\beta S$ process with $\beta \in (0,2)$. If $\alpha \in (0, k - 1/\beta)$ and $p < \beta$, we obtain

$$n^{p(\alpha+1/\beta)-1} V(p,k)_n \xrightarrow{\mathbb{P}} \mathbb{E}[|U|^p]$$

where U is a $S\beta S$ random variable defined via

$$U := c_0 \int_{\mathbb{R}} h_k(s) dL_s.$$

First order asymptotics for power variation

Theorem (cont.)

Assume that $p \geq 1$.

(iii): If $\alpha > k - 1/p$, $p > \beta$ *or* $\alpha > k - 1/\beta$, $p < \beta$, *we deduce*

$$n^{kp-1} V(p,k)_n \xrightarrow{\mathbb{P}} \int_0^1 |F_s^{(k)}|^p ds$$

with

$$F_s^{(k)} = \int_{-\infty}^s g^{(k)}(s-u) dL_u.$$

Summary of first order asymptotics

Theorem

(i): If $\alpha \in (0, k-1/p)$ and $p > \beta$, we obtain as $n \to \infty$

$$n^{\alpha p} V(p,k)_n \xrightarrow{\mathcal{L}-s} |c_0|^p \sum_{m:\ T_m \in [0,1]} |\Delta L_{T_m}|^p V_m.$$

(ii): Assume that L is a $S\beta S$ process with $\beta \in (0,2)$.
If $\alpha \in (0, k-1/\beta)$ and $p < \beta$, we obtain

$$n^{p(\alpha+1/\beta)-1} V(p,k)_n \xrightarrow{\mathbb{P}} \mathbb{E}[|\widetilde{L}_1^{(k)}|^p].$$

(iii): Assume $p \geq 1$. If $\alpha > k-1/p$, $p > \beta$ or $\alpha > k-1/\beta$, $p < \beta$, we deduce

$$n^{kp-1} V(p,k)_n \xrightarrow{\mathbb{P}} \int_0^1 |F_s^{(k)}|^p ds.$$

Remarks

- The above three cases covers all possible cases besides the three boundary cases:

$$\alpha = k - 1/p, \qquad \alpha = k - 1/\beta, \qquad p = \beta.$$

- The rate of convergence in cases (i)–(iii) uniquely identifies the parameters α and β. This might be useful for statistical applications.

- Cases (ii) and (iii) can be extended to a functional convergence. Case (i) is more problematic.

Corollary

Let X be the linear fractional β-stable motion with index H ($\beta < 2$). Suppose that $H < k - 1/p + 1/\beta$ and $p > \beta$. Then

$$n^{-p/\beta + pH} V(p,k)_n \xrightarrow{\mathcal{L}-s} Z = |c_0|^p \sum_{m:\, T_m \in [0,1]} |\Delta L_{T_m}|^p V_m.$$

For $\beta = 2$, X is the fractional Brownian motion and one has

$$n^{-1+pH} V(p,k)_n \xrightarrow{\mathbb{P}} m_{p,k}.$$

Sketch of proof: Case (i)

Step 1: Prove the result when L is a compound Poisson process.

Step 2: Show that small jumps of L are negligible in the limit (we use estimates of [1]).

Step 3: Combine Steps 1 and 2 to conclude the proof.

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[1] B. Rajput and J. Rosiński (1989). Spectral representations of infinitely divisible processes. *Probab. Theory Relat. Fields* 82(3), 451–487.

Sketch of proof: Case (i)

Step 1: Prove the result when L is a compound Poisson process.

- For $x \in \mathbb{R}$ let $\{x\} := \lceil x \rceil - x \in [0,1)$ denote the rounding of x.
- If W is an absolutely continuous random variable then a classical result by Tukey (1938) states that

$$\{nW\} \xrightarrow{d} \mathcal{U}([0,1]) \qquad \text{as } n \to \infty.$$

Second order asymptotics associated with case (ii)

Theorem (B., Lachièze-Rey and Podolskij)

Assume that L is a $S\beta S$ process with $\beta \in (0,2)$.

(a): For $k \geq 2$, $\alpha \in (0, k - 2/\beta)$ and $p < \beta/2$, we obtain

$$\sqrt{n}\left(n^{p(\alpha+1/\beta)-1} V(p,k)_n - \mathbb{E}[|U|^p]\right) \xrightarrow{d} \mathcal{N}(0, v^2).$$

(b): For $k = 1$, $\alpha \in (0, 1 - 1/\beta)$ and $p < \beta/2$, it holds that

$$n^{1-\frac{1}{(1-\alpha)\beta}}\left(n^{p(\alpha+1/\beta)-1} V(p,k)_n - \mathbb{E}[|U|^p]\right) \xrightarrow{d} S_{(1-\alpha)\beta}$$

where $S_{(1-\alpha)\beta}$ is a totally right skewed $(1-\alpha)\beta$-stable random variable with mean zero.

Remark

For the linear fractional stable motion, the self-similarity implies that the random variables in (b) are of form

$$n^{-\frac{1}{(1-\alpha)\beta}} \sum_{i=1}^{n} \left(Y_i - \mathbb{E}[Y_i]\right) \qquad \text{where} \qquad Y_i = |X_i - X_{i-1}|^p.$$

The sequence $(Y_i)_{i\in\mathbb{N}}$ is stationary and each Y_i has finite second moment. By (b) we have that

$$n^{-\frac{1}{(1-\alpha)\beta}} \sum_{i=1}^{n} \left(Y_i - \mathbb{E}[Y_i]\right) \xrightarrow{d} S_{(1-\alpha)\beta},$$

which is very different from what is observed in the classical case with small dependence in $(Y_i)_{i\in\mathbb{N}}$.

Remark

- An idea of proof relies on the identity

$$|x|^p = a_p \int_{\mathbb{R}} \frac{1 - \exp(iux)}{|u|^{1+p}} du, \qquad p \in (0,1).$$

  This identity connects p-th moments of random variables with their characteristic functions.

- Part (b) uses methods of [1] and [2] established for discrete moving averages.

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[1]: Ho and Hsing (1997). Limit theorems for functionals of moving averages. *Ann. Probab.* 25.

[2]: Surgailis (2004): Stable limits of sums of bounded functions of long-memory moving averages with finite variance. *Bernoulli* 10.

Thank you for your attention!

And congratulation Ole!