Markov Renewal Methods in Restart Problems in Complex Systems

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Ole Barndorff-Nielsen

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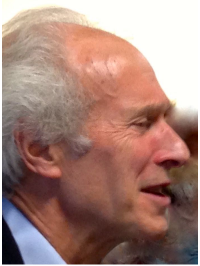
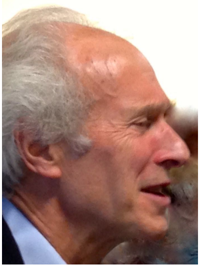


: Nick Bingham

: Ole Barndorff-Nielsen

: Lester Lipsky

: Kishor Trivedi



: Nick Bingham

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- Call centers — 'customer service' by telephone.
  Failures due to broken connection etc.

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What can we say about H given F, G?

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What can we say about H given F, G?

- $\mathbb{E}X =$?? Easy
- $\mathbb{P}(X > x) =$?? Main problem here

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SA–Fiorini-Lipsky-Rolski-Sheahan

Mathematics of Operations Research 2008

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If $L \equiv \ell$: $\mathbb{P}(X > x) \sim C\mathrm{e}^{-\gamma x}$

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Cramér-Lundberg asymptotics: geometric sums, renewal equation

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Theorem

Poisson failures, L gamma, shape α: $\mathbb{P}(X > x) \sim C\dfrac{\log x^{\alpha-1}}{x^{\beta}}$

Theorem

If L has unbounded support: X is heavy-tailed

Why ??

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Why ??

Explanation if $U \stackrel{\mathcal{D}}{=} L$

Theorem

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Why ??

Explanation if $U \stackrel{\mathcal{D}}{=} L$

Total time X to complete $X = U_1^{\#} + \cdots + U_N^{\#} + L$

$U_1^{\#}, \ldots, U_N^{\#}$ failed attempts, N # of restarts

$$N > n \iff U_1 < L, \ldots, U_n < L \iff L = \max(L, U_1, \ldots, U_n)$$

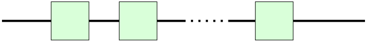
$$\mathbb{P}(N > n) = \frac{1}{n+1}$$

Complex Systems

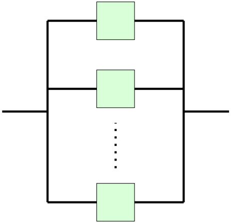
Complex Systems

Classical reliability theory

Series system



Parallel system

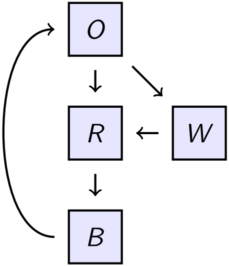


k-out-of-*n*

Repair; cold/warm standby; . . .

2 - out - of - 3 processors, 2 repairmen, warm standby

Processor phases: Operating – δ; Repair – ρ; Booting – β; Waiting



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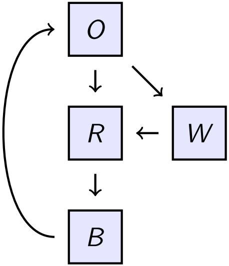
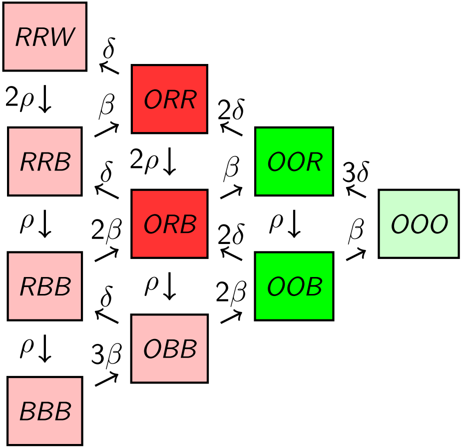
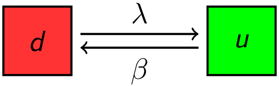




Markov renewal equation. I



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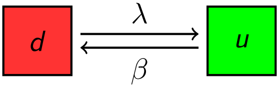

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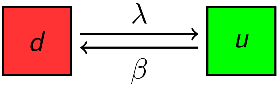

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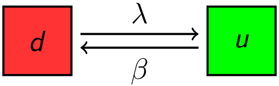

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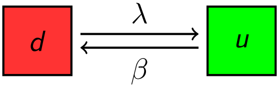

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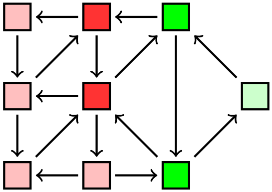
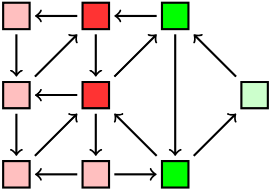
Theorem

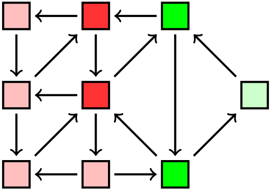
$\mathbb{P}_u(X > x) \sim Ce^{-\gamma x}$ *where* $\gamma > 0$ *solves*

$$1 = \frac{\lambda\beta}{(\lambda-\gamma)(\beta-\gamma)}\left[e^{(\gamma-\beta)\ell} - 1\right] \qquad \textit{and } C = \ldots$$





Memory on task processing carried along to next Markov state !



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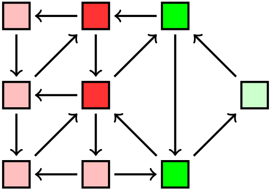
Forgotten when down state entered
$\mathcal{D}$ set of down entrance states (dark red)



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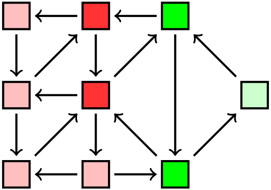

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Markov renewal state space $\mathcal{E} = \mathcal{U} \cup \mathcal{D}$
Sojourn time T_i in $i \in \mathcal{E}$ depends on full generator matrix

Markov renewal equation. II

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$$Z_i(x) = z_i(x) + \sum_{j \in \mathcal{E}} \int_0^x Z_j(x-y)\, F_{ij}(\mathrm{d}y)\,, \quad i \in \mathcal{E}$$

$$F_{du}(\mathrm{d}t) = \mathbb{P}_d(T_d \in \mathrm{d}t, \xi_1 = u) \quad F_{ud}(\mathrm{d}t) = \mathbb{P}_u(T_u \in \mathrm{d}t, t < \ell, \xi_1 = d)$$

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Theorem

Denote by $\mathbf{R}(\alpha)$ *the* $\mathcal{E} \times \mathcal{E}$ *matrix with entries*

$$r_{du}(\alpha) = \mathbb{E}_d\big[\mathrm{e}^{\alpha T_d};\ \xi_1 = u\big]\,, \qquad d \in \mathcal{D}, u \in \mathcal{U}\,,$$

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all other $r_{ij}(\alpha) = 0$. *Assume there exists* $\gamma = \gamma(\ell)$ *such that* $\mathbf{R}(\gamma)$ *is irreducible with* $\mathrm{spr}(\mathbf{R}) = 1$. *Then* $\mathbb{P}_i(X > x) \sim C_i \mathrm{e}^{-\gamma x}$, $x \to \infty$

Markov renewal equation. III

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Needs density f (or "local subexponential behaviour")
Three cases (i) $f << z$, $f \approx z$, $f >> z$

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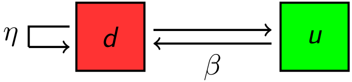
Markov renewal case: SA - Thøgersen 2015

Heavy-tailed example

Ideal repair time R (random); rate η failures of repair



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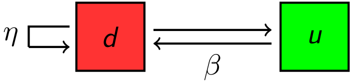

Actual repair time: vanilla Restart

If R is Gamma: $\mathbb{P}(X > x) \ \sim \ C \dfrac{\log^{\alpha-1} x}{x^{\mu}}$

Random task time L

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Random task time L

$$\mathbb{P}_i(X > x) = \int_0^\infty \mathbb{P}_i(X > x \mid L = \ell)\, \mathbb{P}(L \in \mathrm{d}\ell)\,.$$

Know $\mathbb{P}_i(X > x \mid L = \ell) \sim C_i \mathrm{e}^{-\gamma(\ell)x}$ (with light T_d tails)
$\gamma(\ell)$ solution of $1 = \mathrm{spr}\big(\mathbf{R}(\gamma, \ell)\big)$

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Corollary

If the task length L has unbounded support, the distribution of the total task time X is heavy-tailed in the sense that $\mathrm{e}^{\delta x}\mathbb{P}(X > x) \to \infty$ for all $\delta > 0$.

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More precise asymptotics?
Needs asymptotics of $\gamma(\ell)$

Theorem

Assume that for some function $\varphi(\ell)$ it holds that

$$\mathbb{P}\big(T_u > \ell,\, \xi_1 = d\big) \sim k_{ud}\varphi(\ell)$$

as $\ell \to \infty$ for some set of constants such that $k_{ud} > 0$ for at least one pair $u \in \mathcal{U}$, $d \in \mathcal{D}$. Then

$$\gamma(\ell) \sim \mu\,\varphi(\ell) \quad \text{as } \ell \to \infty, \qquad \text{where} \quad \mu = \frac{\sum\limits_{u\in\mathcal{U},d\in\mathcal{D}} \pi_u k_{ud}}{\sum\limits_{i\in\mathcal{U}\cup\mathcal{D}} \pi_i \mathbb{E}_i T_i}$$

and $\pi = (\pi_i)_{i\in\mathcal{U}\cup\mathcal{D}}$ is the stationary distribution of the Markov chain ξ, that is, the invariant probability vector for the matrix $\mathbf{P} = \mathbf{R}(0,\infty)$.

Proof ??

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$\gamma(\ell)$ solves $1 = \text{spr}\big(\mathbf{R}(\gamma,\ell)\big)$

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1) Perturbation theory $\mathbf{R}(0, \infty) = \mathbf{P}$

$\text{spr}\big(\mathbf{R}(\gamma, \ell)\big) = \text{spr}(\mathbf{P}) +$?? $= 1 +$??

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$$r_{du}(\alpha) = \mathbb{E}_d\left[\mathrm{e}^{\alpha T_d};\ \xi_1 = u\right], \qquad d \in \mathcal{D},\ u \in \mathcal{U},$$
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1) Perturbation theory $\mathbf{R}(0, \infty) = \mathbf{P}$
$\mathrm{spr}\big(\mathbf{R}(\gamma, \ell)\big) = \mathrm{spr}(\mathbf{P}) + ?? = 1 + ??$

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3) Bare-hand (but Perron-Frobenius theory key tool)

Back to Markov set-up

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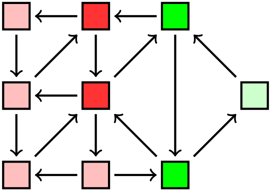

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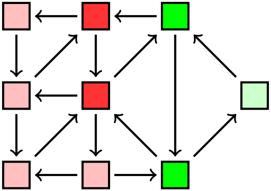
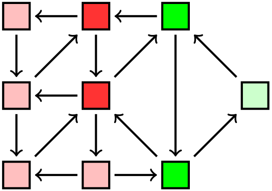


Markov renewal up states $\mathcal{U}$: two dark green
Markov renewal down states $\mathcal{D}$: two dark red

Back to Markov set-up



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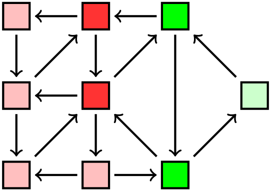
Markov up states $\mathcal{U}^*$: all three green
Markov down states $\mathcal{D}^*$: all six red

Back to Markov set-up



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Root $\gamma(\ell)$ for Markov renewal equation depends on full Markov model

Theorem

Assume that $\gamma = \gamma(\ell)$ makes the spectral radius of the matrix $\mathbf{R}(\gamma, \ell)$ equal to 1, where $\mathbf{R}(\gamma, \ell)$ is the matrix

	$\mathcal{U}$	$\mathcal{D}$
$\mathcal{U}$	0	$\widehat{F}_{ud}[\gamma]$
$\mathcal{D}$	$\widehat{F}_{du}[\gamma]$	0

$$\widehat{F}_{ud}[\gamma] = \int_0^\ell \mathrm{e}^{\gamma t} F_{ud}(\mathrm{d}t)$$

$$\widehat{F}_{du}[\gamma] = \int_0^\infty \mathrm{e}^{\gamma t} F_{du}(\mathrm{d}t)$$

Then $\gamma(\ell) \sim \mu \mathrm{e}^{-\delta \ell}$ as $\ell \to \infty$, where $-\delta$ is the largest eigenvalue of $\mathbf{Q}_{\mathcal{U}^*\mathcal{U}^*}$ in the block-partitioning

	$\mathcal{U}^*$	$\mathcal{D}^*$
$\mathcal{U}^*$	$\mathbf{Q}_{\mathcal{U}^*\mathcal{U}^*}$	$\mathbf{Q}_{\mathcal{U}^*\mathcal{D}^*}$
$\mathcal{D}^*$	$\mathbf{Q}_{\mathcal{D}^*\mathcal{U}^*}$	$\mathbf{Q}_{\mathcal{D}^*\mathcal{D}^*}$

of the full generator $\mathbf{Q}$ and μ involves $\mathbf{Q}_{\mathcal{U}^*\mathcal{U}^*}^{-1}$, $\mathbf{Q}_{\mathcal{D}^*\mathcal{D}^*}^{-1}$ and further Perron-Frobenius characteristics of $\mathbf{Q}$.

General approach for random task time L

$f(\ell)$ density of ℓ

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$$\mathbb{P}_i(X > x) \;=\; \int_0^\infty \mathbb{P}_i(X > x \,|\, L = \ell)\, f(\ell)\, \mathrm{d}\ell$$

General approach for random task time L

$f(\ell)$ density of ℓ

$$\begin{aligned}\mathbb{P}_i(X > x) &= \int_0^\infty \mathbb{P}_i(X > x \mid L = \ell)\, f(\ell)\, \mathrm{d}\ell \\ &\sim \int_0^\infty C_i(\ell) \exp\{-\gamma(\ell)x\}\, f(\ell)\, \mathrm{d}\ell\end{aligned}$$

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Most often OK; now purely analytical problem.

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Most often OK; now purely analytical problem.

Corollary

Assume failures are Poisson(δ) *(or* $\varphi(\ell) = \mathrm{e}^{-\delta\ell}$*) and that* F *is gamma-like in the sense that* $f(\ell) \sim c_F \ell^{\alpha-1}\mathrm{e}^{-\lambda\ell}$, $\ell \to \infty$*. Then*

$$\mathbb{P}_i(X > x) \sim \frac{C_i^* \Gamma(\lambda/\delta)}{\delta^{\alpha+\lambda/\delta}} \frac{\log^{\alpha-1} x}{x^{\lambda/\delta}} \text{ as } x \to \infty.$$

4×4 table of examples of rough $\mathbb{P}(X > x)$ asymptotics
each of $f(\ell), \varphi(\ell)$ LT Weibull; exponential; HT Weibull; power

$f(\ell)$ / $\varphi(\ell)$	$e^{-\ell^2}$	$e^{-\ell}$	$e^{-\ell^{1/2}}$	$\frac{1}{\ell^\alpha}$
$e^{-\ell^2}$	$\frac{1}{x}$	$e^{-\log^{1/2} x}$	$e^{-\log^{1/4} x}$	$\frac{1}{\log^{\alpha/2} x}$
$e^{-\ell}$	$e^{-\log^2 x}$	$\frac{1}{x}$	$e^{-\log^{1/2} x}$	$\frac{1}{\log^{\alpha} x}$
$e^{-\ell^{1/2}}$	$e^{-\log^4 x}$	$e^{-\log^2 x}$	$\frac{1}{x}$	$\frac{1}{\log^{2\alpha} x}$
$\frac{1}{\ell^\alpha}$	$e^{-x^{\frac{2}{2+\alpha}}}$	$e^{-x^{\frac{1}{1+\alpha}}}$	$e^{-x^{\frac{1/2}{1/2+\alpha}}}$	$\frac{1}{x}$

Constants omitted $e^{-c\log^{1/2} x}$; $\dfrac{1}{x} = e^{-\log x}$
Logarithmic asymptotics
In some corners even $\log\log$ asymptotics

Tauberian theorem

$$\mathbb{P}_i(X > x) \sim \int_0^\infty D_i(\ell) \exp\{-\mu\varphi(\ell)x\} f(\ell)\, \mathrm{d}\ell$$

Theorem

Define $\overline{\varphi}_I(t) = \int_t^\infty \varphi(y)\, \mathrm{d}y$ *and assume*

$$f(t) = \varphi(t)\overline{\varphi}_I(t)^{\beta-1} L_0(\overline{\varphi}_I(t))$$

where $L_0(s)$ *is slowly varying at* $s = 0$. *Then*

$$\mathbb{P}_i(X > x) \sim D_i^* \frac{\Gamma(\beta)}{\mu^\beta} \frac{L_0(1/x)}{x^\beta}, \quad x \to \infty .$$

Rare events approach

$\mathbb{P}(X(\ell) > x)$: sofar first $x \to \infty$, then $\ell \to \infty$.
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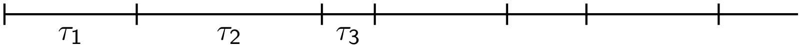

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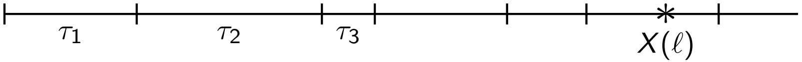

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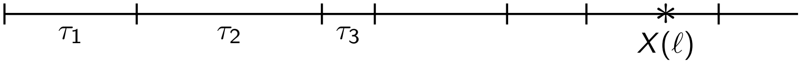

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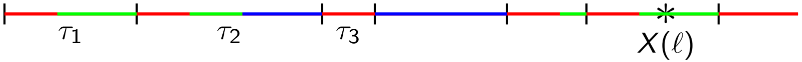
Cycle start here: entrance to reference Markov renewal state



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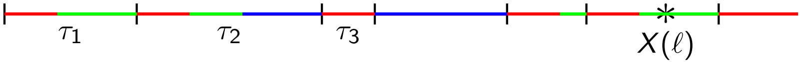

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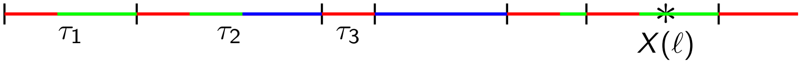

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Theorem

$$\mathbb{E}X(\ell) \sim \frac{1}{\gamma(\ell)} \sim \frac{1}{\mu\varphi(\ell)}, \quad \ell \to \infty.$$

Non-exponential distributions

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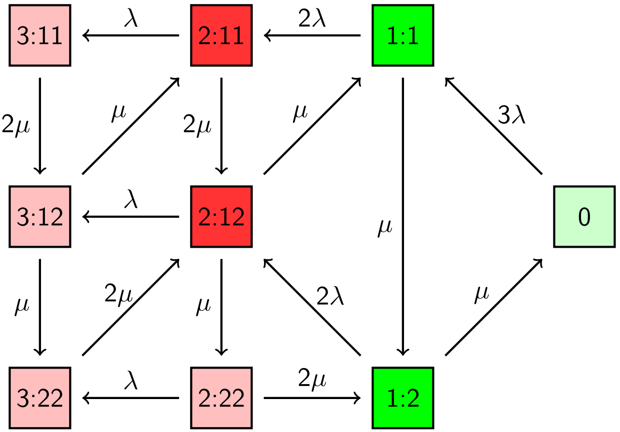
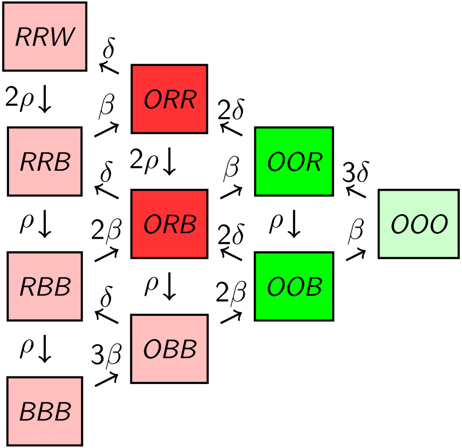


Figure: E_2 repair times, 2-out-of-3

Variable rates

Variable rates

2 - out - of - 3 processors, 2 repairmen, warm standby



Rate ρ of each processor

Total rate 3ρ in OOO, 2ρ in OOO

Task processed at rate $\rho_u(t)$ in $u \in \mathcal{U}$

completion at time $\inf\left\{s > 0 : \int_0^s \rho_u(t)\,\mathrm{d}t \geq \ell\right\}$ if $< T_u$

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Theorem

Consider the Markov model with $\rho_u(t) = r_{J(t)}$. Then $(*)$ becomes

$$\mathbf{e}_u^T \Big(\mathbf{I} - \exp\{\boldsymbol{\Delta}_{\mathbf{r}}^{-1}(\mathbf{Q}_{\mathcal{U}^*\mathcal{U}^*} + \alpha \mathbf{I})\ell\}\Big)(-\mathbf{Q}_{\mathcal{U}^*\mathcal{U}^*} - \alpha \mathbf{I})^{-1}\mathbf{Q}_{\mathcal{U}^*\mathcal{D}^*}\mathbf{e}_d\,.$$

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Formulas in terms of

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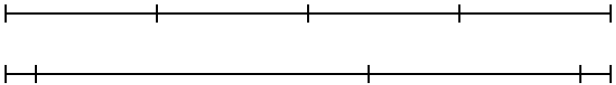
Or T_u PH

Fragmentation

Fragmentation

K parts, $L = L_1 + \cdots + L_K$

Equidistant, footer and header, etc.



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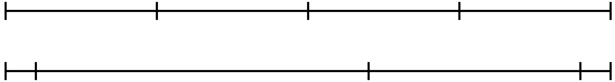

Parallel computing:
$X = \max(X_1, \ldots, X_K)$
E.g. Monte Carlo, $R = R_1 + \cdots + R_K$ replications, $R_k = R/K$

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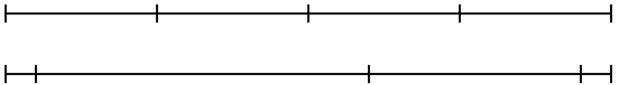

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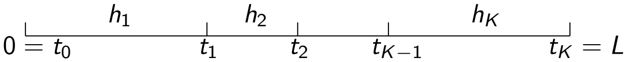
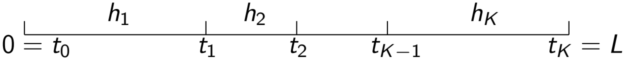
Checkpointing:
$X = X_1 + \cdots + X_K$
Previous part needs completion

Checkpoint modeling

Checkpoint modeling

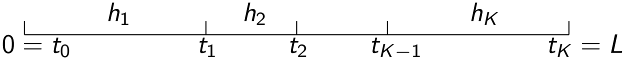


Checkpoint modeling



A: L deterministic, $L \equiv \ell$, checkpoints deterministic and equally spaced,
$t_1 = h/K,\ t_2 = 2h/K, \ldots,\ t_{K-1} = (K-1)h/K.$

Checkpoint modeling



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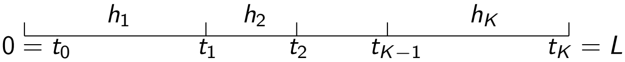
B: L deterministic, checkpoints deterministic but not equally spaced.

Checkpoint modeling



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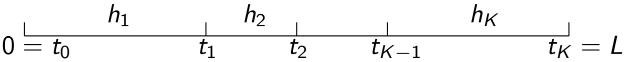

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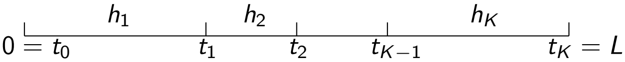

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  NB: ignores cost of checkpointing
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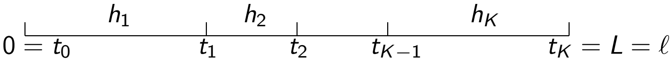
Which scheme is best given K?

Take L deterministic, $L \equiv \ell$ (Models A,B,C).



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Conjecture: main contribution to X comes from longest T_k. I.e., it should be best to take $T_k = T/K$ (Model A).

Ordering of checkpointing models A,B,C

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Another possibility:
(ii) $\mathbb{P}(X_{\mathrm{A}} > x) \le \mathbb{P}(X_{\mathrm{B}} > x)$ and $\mathbb{P}(X_{\mathrm{A}} > x) \le \mathbb{P}(X_{\mathrm{C}} > x)$
for all large x

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Intuitive explanation:
Placing checkpoint at 1, failure mechanism starts afresh then.
I.e., the failure rate becomes $g_1(u) > 0$ instead of $g_2(u) = 0$.

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Main case: Poisson failures.

Limit theorems models A,B,C. Comparison with R

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RESTART comparison: $\mathbb{P}(X_R > x) \ \sim \ \dfrac{C_{\mathrm{R}}}{x^{\lambda/\mu}}$
Reduction from power tail to exponential tail.

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RESTART comparison: $\mathbb{P}(X_R > x) \ \sim \ C_\mathrm{R} \exp\{-\theta \log \log x\}$
Heavier than any power; reduction to power tail.

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Assume $t_k = kT/K$ *and that* F *is exponential*(λ). *Then*

$$\mathbb{P}(X_{\mathrm{E}} > x) \sim \frac{C_{\mathrm{E}})}{x^{\lambda/\mu+K-1}}$$

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RESTART comparison: $\mathbb{P}(X_R > x) \sim \dfrac{C_{\mathrm{R}}}{x^{\lambda/\mu}}$
Still power tail, but each checkpoint improves the power by 1.

Thanks

Thanks

to all co-authors

Thanks

to all co-authors

to the audience for your patience

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and to Nick