Ambit Stochastics

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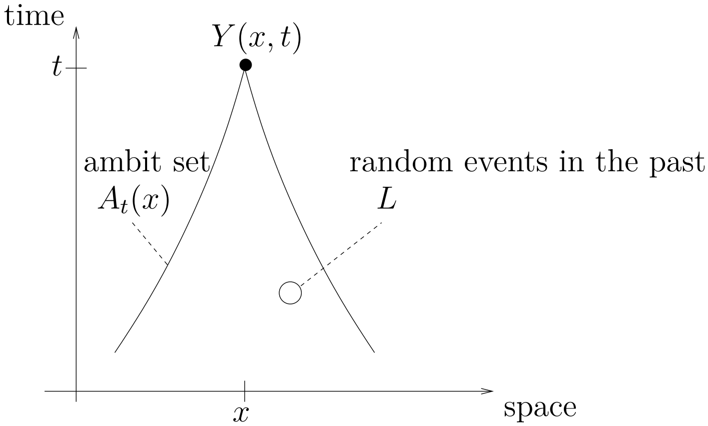
Outline

- Framework
- Ambit fields of exponential type
- Application: Cascades
- Application: Turbulent time series
- Application: Growth
- Outlook: Particle transport

intuitive approach: causality cone



Framework

Additive type:

$$Y(x,t) = \int_{A_t(x)} \cdots \mathrm{d}L$$

Exponential type:

$$Y(x,t) = \exp\left\{ \int_{A_t(x)} \cdots \mathrm{d}L \right\}$$

Framework

Ambit field:

$$Y_t(x) = \int_{A_t(x)} g(t,s,x,y) b_s(y)\, L(ds\, dy) + \int_{D_t(x)} q(t,s,x,y) a_s(y)\, ds\, dy$$

- $A_t(x)$,$D_t(x)$: ambit sets
- g, q: deterministic weight functions
- b: volatility/intermittency field
- a: drift field
- L: Lévy basis

Framework

Stationary and homogeneous ambit field:

$$Y_t(x) = \int_{A_t(x)} g(t-s, |x-y|) b_s(y) \, L(ds\, dy) + \int_{D_t(x)} q(t-s, |x-y|) a_s(y) \, ds\, dy$$

- $A_t(x) = (x,t) + A_0$: translational invariant ambit set
- $D_t(x) = (x,t) + D_0$: translational invariant ambit set
- b: stationary and homogeneous volatility/intermittency field
- a: stationary and homogeneous drift field
- L: homogeneous Lévy basis

Ambit fields of exponential type

Special case of a stationary and homogeneous ambit field of exponential type:

$$Y_t(x) = \exp\left\{\int_{A_t(x)} \mathrm{d}L\right\} = \exp\left\{L(A_t(x))\right\}$$

- L: homogeneous Lévy basis
- $A_t(x) = (x,t) + A \subset \mathbb{R}^2$: translational invariant ambit set

Ambit fields of exponential type

Two-point correlators:

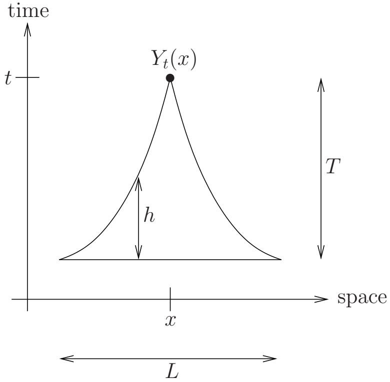
$$c_{n_1,n_2}(x,t) = \frac{\mathrm{E}\left\{Y_t(x)^{n_1} Y_0(0)^{n_2}\right\}}{\mathrm{E}\left\{Y_t(x)^{n_1}\right\}\mathrm{E}\left\{Y_0(0)^{n_2}\right\}}$$
$$= \exp\left\{\overline{\mathrm{K}}[n_1,n_2]\,|A_t(x)\cap A_0(0)|\right\}$$

Covariogram problem:

Reconstruction of $A \subset \mathbb{R}^n$ from the knowledge of $|((x,t)+A) \cap A|$ for all (x,t)

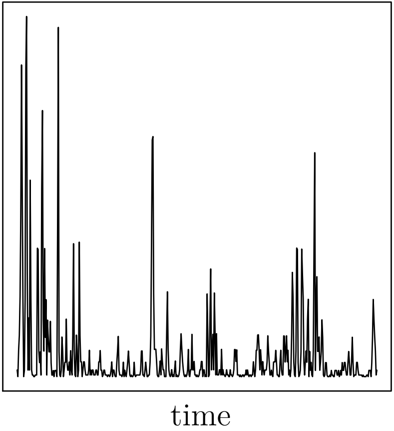
Boundary function h:

Determined from equal time or equal position two-point correlators



Application: Cascades

Energy dissipation in a turbulent flow



Application: Cascades

Correlators of a cascade process Y_t:

$$c_{n_1,n_2}(t) = \frac{\mathrm{E}\left\{Y_t^{n_1} Y_0^{n_2}\right\}}{\mathrm{E}\left\{Y_t^{n_1}\right\} \mathrm{E}\left\{Y_0^{n_2}\right\}} \propto t^{-\xi(n_1,n_2)}$$

for a certain range of $t \in [t_l, t_u]$.

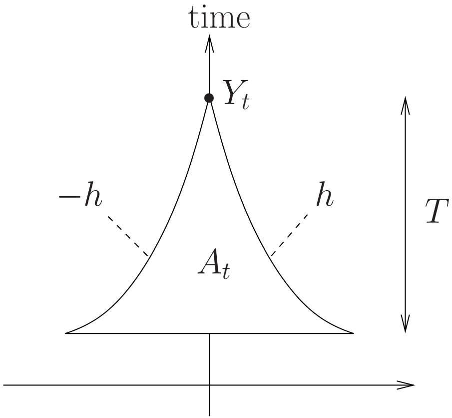
Ambit field of exponential type:

$$Y_t = \exp\left\{\int_{A_t} \mathrm{d}L\right\} = \exp\left\{L(A_t)\right\}$$

- L: homogeneous Lévy basis
- $A_t = (0,t) + A \subset \mathbb{R}^2$: translational invariant ambit set

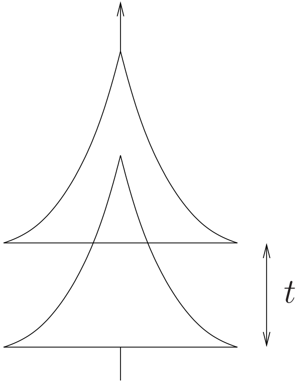
Application: Cascades

Ambit set A_t:



Application: Cascades

Correlators: $c_{n_1,n_2}(t) = \exp\left\{\overline{\mathrm{K}}[n_1,n_2]\,|A_t \cap A_0|\right\}$



Modeling the energy dissipation: input

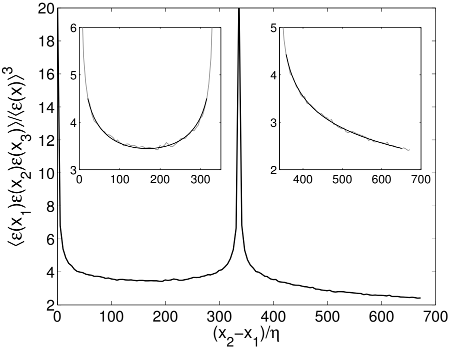
- marginal distribution
- two-point correlators

Application: Cascades

Modeling the energy dissipation: output

- scaling of correlators $c_{p,q}$
- unconditional and conditional multiplier distributions
- multi-point correlators
- $\dots$

Modeling the energy dissipation: three point correlators



Application: Turbulent time series

Main component of the turbulent velocity vector: Brownian semistationary processes

$$Y_t = \int_{-\infty}^{t} g(t-s)\sigma_s \mathrm{d}B_s + \beta \int_{-\infty}^{t} g(t-s)\sigma_s^2 \mathrm{d}s$$

- g: deterministic kernel
- σ: stochastic volatility
- β: constant

Modeling the turbulent velocity vector: input

- cascade model for σ
- second order structure function of velocity increments
- skewness of velocity increments

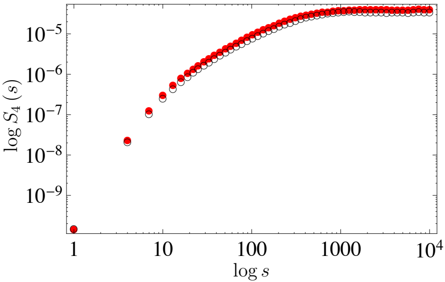
Application: Turbulent time series

Stylized features: output

- distribution of velocity increments
- aggregational Gaussianity
- scaling of higher order structure functions
- statistics of the energy dissipation
- statistics of the Kolmogorov variable

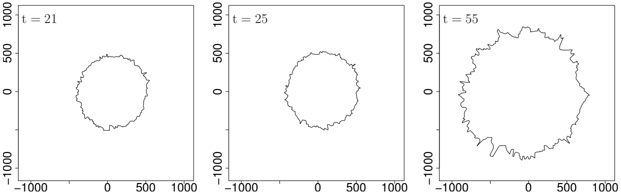
Fourth order structure function:

$$S_4(s) = \mathrm{E}\left\{(Y_{t+s} - Y_t)^4\right\}$$

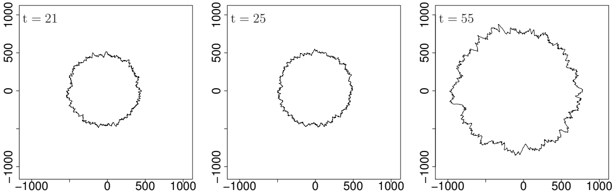


Application: Growth

Star shaped tumor growth



Ambit model



Application: Growth

Normalized radius function

$$r_t(\Phi) = \frac{R_t(\Phi)}{\mathrm{E}\left\{R_t(\Phi)\right\}}$$

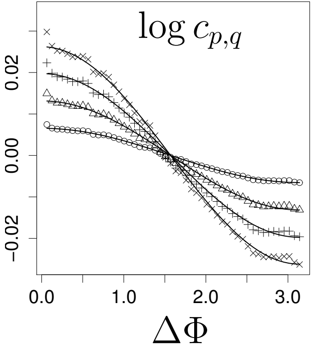
Ambit field of exponential type

$$r_t(\Phi) = \exp\left\{a(t)\int_{A_t^{(1)}(\Phi)} \cos(\Phi - \Phi')\mathrm{L}(\mathrm{d}t'\mathrm{d}\Phi') + h(t)\int_{A_t^{(2)}(\Phi)} \mathrm{L}(\mathrm{d}t'\mathrm{d}\Phi')\right\}$$

Application: Growth

Equal time correlators

$$c_{p,q}(\Phi, t; \Phi + \Delta\Phi, t) = \frac{\mathrm{E}\{R_t(\Phi)^p R_t(\Phi{+}\Delta\Phi)^q\}}{\mathrm{E}\{R_t(\Phi)^p\}\mathrm{E}\{R_t(\Phi{+}\Delta\Phi)^q\}}$$



Outlook: Particle transport

Ambit framework:

- vector models in 3-dimensional space and time
- data-driven modeling
- flexible and mathematically tractable
- non-stationarity and non-homogeneity

Outlook: Particle transport

Modeling the turbulent background field:

- as initial values for direct numerical simulations
- input to dynamical equations

Outlook: Particle transport

Direct modeling of the active random forces:

- modeling of dynamically active forces
- data-driven: stylized features of dynamically active forces

Outlook: Particle transport

Direct modeling of particle dynamics:

- modeling the velocity/path of a particle or the density of an ensemble of particles in a turbulent flow
- data-driven: stylized features of particle transport