Stochastic processes for symmetric space-time fractional diffusion

Gianni PAGNINI
IKERBASQUE Research Fellow
BCAM - Basque Center for Applied Mathematics
Bilbao, Basque Country - Spain
gpagnini@bcamath.org

Main Goal of the Research

Classical/normal diffusion is described by a Gaussian process.

- Diffusion/Brownian motion

Is it possible to describe **anomalous diffusion/fractional diffusion** with a stochastic process that is still based on a Gaussian process?

- Time-fractional diffusion/grey Brownian motion (Schneider 1990, 1992)
- Erdélyi–Kober fractional diffusion/generalized grey Brownian motion (Mura 2008, Pagnini 2012)
- Space-time fractional diffusion/present stochastic process

The Space-Time Fractional Diffusion Equation

$$ {}_x\mathcal{D}_\theta^\alpha \, K_{\alpha,\beta}^\theta(x;t) = {}_t\mathcal{D}_*^\beta \, K_{\alpha,\beta}^\theta(x;t) \,, \quad K_{\alpha,\beta}^\theta(x;0) = \delta(x) \,, \tag{1} $$

$$ 0 < \alpha \le 2 \,, \quad |\theta| \le \min\{\alpha, 2-\alpha\} \,, \tag{2a} $$

$$ 0 < \beta \le 1 \quad \text{or} \quad 1 < \beta \le \alpha \le 2 \,. \tag{2b} $$

${}_x\mathcal{D}_\theta^\alpha$: Riesz–Feller space-fractional derivative

$$ \mathcal{F}\left\{ {}_x\mathcal{D}_\theta^\alpha \, f(x);\kappa \right\} = -|\kappa|^\alpha \, \mathrm{e}^{i(\mathrm{sign}\,\kappa)\theta\pi/2} \, \widehat{f}(\kappa) \,. \tag{3} $$

${}_t\mathcal{D}_*^\beta$: Caputo time-fractional derivative

$$ \mathcal{L}\left\{ {}_t\mathcal{D}_*^\beta \, f(t);s \right\} = s^\beta \, \widetilde{f}(s) - \sum_{j=0}^{m-1} s^{\beta-1-k} \, f^{(j)}(0^+) \,, \tag{4} $$

with $m-1 < \beta \le m$ and $m \in N$.

The Green Function $K^\theta_{\alpha,\beta}(x;t)$

Self-similarly law

$$K^\theta_{\alpha,\beta}(x;t) = t^{-\beta/\alpha}\, K^\theta_{\alpha,\beta}\left(\frac{x}{t^{\beta/\alpha}}\right) . \tag{5}$$

Symmetry relation

$$K^\theta_{\alpha,\beta}(-x;t) = K^{-\theta}_{\alpha,\beta}(x;t)\,, \tag{6}$$

which allows the restriction to $x \geq 0$.

Mellin–Barnes integral representation

$$K^\theta_{\alpha,\beta}(x;t) = \frac{1}{\alpha x}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\Gamma(\frac{q}{\alpha})\Gamma(1-\frac{q}{\alpha})\Gamma(1-q)}{\Gamma(1-\frac{\beta}{\alpha}q)\Gamma(\rho\, q)\Gamma(1-\rho\, q)} \left(\frac{x}{t^{\beta/\alpha}}\right)^q dq\,, \tag{7}$$

where $\rho = (\alpha-\theta)/(2\,\alpha)$ and c is a suitable real constant.

Mainardi, Luchko, Pagnini Fract. Calc. Appl. Anal. 2001

Special Cases: $x > 0$

$$K_{2,1}^{0}(x;t) = \frac{1}{\sqrt{4\pi t}}\,\mathrm{e}^{-x^2/(4t)} = G(x;t) = t^{-1/2} G\left(\frac{x}{t^{1/2}}\right) , \qquad (8)$$

$$K_{\alpha,1}^{\theta}(x;t) = L_{\alpha}^{\theta}(x;t) = t^{-1/\alpha} L_{\alpha}^{\theta}\left(\frac{x}{t^{1/\alpha}}\right) , \qquad (9)$$

$$K_{2,\beta}^{0}(x;t) = \frac{1}{2} M_{\beta/2}(x;t) = \frac{1}{2} t^{-\beta/2} M_{\beta/2}\left(\frac{x}{t^{\beta/2}}\right) , \qquad (10)$$

$$K_{\alpha,\alpha}^{\theta}(x;t) = \frac{t^{-1}}{\pi} \frac{(x/t)^{\alpha-1} \sin[\frac{\pi}{2}(\alpha-\theta)]}{1 + 2(x/t)^{\alpha} \cos[\frac{\pi}{2}(\alpha-\theta)] + (x/t)^{2\alpha}} , \qquad (11)$$

$$K_{2,2}^{0}(x;t) = \frac{1}{2}\delta(x-t) . \qquad (12)$$

Integral Representation Formulae for $K^{\theta}_{\alpha,\beta}(x;t)$

If $x > 0$ then

$$K^{\theta}_{\alpha,\beta}(x;t) = \int_0^{\infty} L^{\theta}_{\alpha}(x;\tau) L^{-\beta}_{\beta}(t;\tau) \frac{t}{\tau\beta}\, d\tau\,, \quad 0 < \beta \leq 1\,, \qquad (13)$$

$$K^{\theta}_{\alpha,\beta}(x;t) = \int_0^{\infty} L^{\theta}_{\alpha}(x;\tau)\, M_{\beta}(\tau;t)\, d\tau\,, \quad 0 < \beta \leq 1\,, \qquad (14)$$

$$K^{\theta}_{\alpha,\beta}(x;t) = \int_0^{\infty} K^{\theta}_{\alpha,\alpha}(x;\tau)\, M_{\beta/\alpha}(\tau;t)\, d\tau\,, \quad 0 < \beta/\alpha \leq 1\,. \qquad (15)$$

Mainardi, Luchko, Pagnini Fract. Calc. Appl. Anal. 2001

Supplementary Results

From formulae (13) and (14) it follows that

$$\frac{1}{\tau^{1/\beta}} L_\beta^{-\beta}\left(\frac{t}{\tau^{1/\beta}}\right) = \frac{\tau\,\beta}{t^{1+\beta}} M_\beta\left(\frac{\tau}{t^\beta}\right), \quad 0 < \beta \leq 1\,, \quad \tau, t > 0\,. \tag{16}$$

From formulae (8) and (9)

$$K_{2,1}^0(x;t) = G(x;t) = L_2^0(x;t)\,. \tag{17}$$

From formulae (8) and (10)

$$K_{2,1}^0(x;t) = G(x;t) = \frac{1}{2} M_{1/2}(x;t)\,. \tag{18}$$

Supplementary Results

$$L_\alpha^\theta(x;t) = \int_0^\infty L_\eta^\omega(x;\xi) L_\nu^{-\nu}(\xi;t)\, d\xi\,, \quad \alpha = \eta\nu\,, \quad \theta = \omega\nu\,, \quad (19)$$

$$0 < \alpha \le 2\,, \quad |\theta| \le \min\{\alpha, 2-\alpha\}\,,$$

$$0 < \eta \le 2\,, \quad |\omega| \le \min\{\eta, 2-\eta\}\,, \quad 0 < \nu \le 1\,.$$

In particular it holds

$$L_\alpha^0(x;t) = \int_0^\infty L_2^0(x;\xi) L_{\alpha/2}^{-\alpha/2}(\xi;t)\, d\xi \quad (20)$$

$$= \int_0^\infty G(x;\xi) L_{\alpha/2}^{-\alpha/2}(\xi;t)\, d\xi\,. \quad (21)$$

Mainardi, Pagnini, Gorenflo Fract. Calc. Appl. Anal. 2003

Supplementary Results

$$M_\nu(x;t) = \int_0^\infty M_\eta(x;\xi) M_\beta(\xi;t)\, d\xi\,, \quad \nu = \eta\beta\,, \qquad (22)$$

$$0 < \nu, \eta, \beta \le 1\,.$$

In particular it holds

$$M_{\beta/2}(x;t) = 2\int_0^\infty M_{1/2}(x;\xi) M_\beta(\xi;t)\, d\xi \qquad (23)$$

$$= 2\int_0^\infty G(x;\xi) M_\beta(\xi;t)\, d\xi\,. \qquad (24)$$

Mainardi, Pagnini, Gorenflo Fract. Calc. Appl. Anal. 2003

New Integral Representation Formula for $K^{\theta}_{\alpha,\beta}(x;t)$

Consider formula (14), i.e. $K^{\theta}_{\alpha,\beta}(x;t) = \int_0^\infty L^{\theta}_{\alpha}(x;\tau) M_\beta(\tau;t)\, d\tau$, then and by using (19) it follows

$$\begin{aligned}
K^{\theta}_{\alpha,\beta}(x;t) &= \int_0^\infty \left\{ \int_0^\infty L^{\omega}_{\eta}(x;\xi) L^{-\nu}_{\nu}(\xi;t)\, d\xi \right\} M_\beta(\tau,t)\, d\tau \\
&= \int_0^\infty L^{\omega}_{\eta}(x;\xi) \left\{ \int_0^\infty L^{-\nu}_{\nu}(\xi;t) M_\beta(\tau;t)\, d\tau \right\} d\xi \\
&= \int_0^\infty L^{\omega}_{\eta}(x;\xi)\, K^{-\nu}_{\nu,\beta}(\xi;t)\, d\xi\,, \quad \alpha = \eta\nu\,, \quad \theta = \omega\nu\,,
\end{aligned}$$

$$0 < \alpha \le 2\,, \quad |\theta| \le \min\{\alpha, 2-\alpha\}\,, \quad 0 < \beta \le 1\,,$$
$$0 < \eta \le 2\,, \quad |\omega| \le \min\{\eta, 2-\eta\}\,, \quad 0 < \nu \le 1\,.$$

New Integral Representation Formula for $K_{\alpha,\beta}^{\theta}(x;t)$

$$K_{\alpha,\beta}^{\theta}(x;t) = \int_0^{\infty} L_{\eta}^{\omega}(x;\xi)\, K_{\nu,\beta}^{-\nu}(\xi;t)\, d\xi\,, \tag{25}$$

$$0 < x < +\infty\,, \quad \alpha = \eta\nu\,, \quad \theta = \omega\nu\,,$$

$$0 < \alpha \le 2\,, \quad |\theta| \le \min\{\alpha, 2-\alpha\}\,, \quad 0 < \beta \le 1\,,$$

$$0 < \eta \le 2\,, \quad |\omega| \le \min\{\eta, 2-\eta\}\,, \quad 0 < \nu \le 1\,.$$

In the spatial symmetric case, i.e. $\eta = 2$ and $\omega = 0$ such that $L_2^0 \equiv G$, hence $\nu = \alpha/2$ and $\theta = 0$ and formula (25) gives

$$K_{\alpha,\beta}^{0}(x;t) = \int_0^{\infty} G(x;\xi)\, K_{\alpha/2,\beta}^{-\alpha/2}(\xi;t)\, d\xi\,. \tag{26}$$

$$-\infty < x < +\infty\,, \quad 0 < \alpha \le 2\,, \quad 0 < \beta \le 1\,.$$

Special Cases: $\alpha = 2$ and $0 < \beta \leq 1$

From (14) it results that

$$\begin{aligned} K_{1,\beta}^{-1}(\xi;t) &= \int_0^\infty L_1^{-1}(\xi;\tau)\, M_\beta(\tau;t)\, d\tau \\ &= \int_0^\infty \delta(\xi-\tau)\, M_\beta(\tau;t)\, d\tau = M_\beta(\xi;t)\,, \quad (27) \end{aligned}$$

finally by using (24)

$$\begin{aligned} K_{2,\beta}^{0}(x;t) &= \int_0^\infty G(x;\xi)\, K_{1,\beta}^{-1}(\xi;t)\, d\xi \quad (28) \\ &= \int_0^\infty G(x;\xi)\, M_\beta\, d\xi = \frac{1}{2} M_{\beta/2}(x;t)\,. \quad (29) \end{aligned}$$

Special Cases: $0 < \alpha \le 2$ and $\beta = 1$

From (14) it results that

$$
\begin{aligned}
K_{\alpha/2,1}^{-\alpha/2}(\xi;t) &= \int_0^\infty L_{\alpha/2}^{-\alpha/2}(\xi;\tau)\, M_1(\tau;t)\, d\tau \\
&= \int_0^\infty L_{\alpha/2}^{-\alpha/2}(\xi;\tau)\, \delta(\tau - t)\, d\tau = L_{\alpha/2}^{-\alpha/2}(\xi;t)\,, \quad (30)
\end{aligned}
$$

finally by using (21)

$$
\begin{aligned}
K_{\alpha,1}^{0}(x;t) &= \int_0^\infty G(x;\xi)\, K_{\alpha/2,1}^{-\alpha/2}(\xi;t)\, d\xi \quad (31) \\
&= \int_0^\infty G(x;\xi)\, L_{\alpha/2}^{-\alpha/2}(\xi;t)\, d\xi = L_{\alpha}^{0}(x;t)\,. \quad (32)
\end{aligned}
$$

Definitions in Stochastic Process Theory

Stationary stochastic processes: A stochastic process $X(t)$ whose one-time statistical characteristics do not change in the course of time t, i.e. they are invariant relative to the time shifing $t \to t + \tau$ for any fixed value of τ, and two-time statistics (e.g. autocorrelation) depend solely on the elapsed time τ,

$$\langle X(t)^n \rangle = \langle X^n(t+\tau) \rangle = C_n \,, \quad \langle X(t) X(t+\tau) \rangle = R(\tau) \,. \qquad (33)$$

Definitions in Stochastic Process Theory

Stochastic processes with stationary increments: A stochastic process $X(t)$ such that the statistical characteristics of its increments $\Delta X(t) = X(t) - X(t+\tau)$ do not vary in the course of time t, i.e. they are invariant relative to the time shifting $t \to t + s$ for any fixed value of s,

$$\langle \Delta X(t) \rangle = 0 \,, \quad \langle (\Delta X(t))^2 \rangle = 2\,[C_2 - R(\tau)] \,. \qquad (34)$$

Because $R(\tau = 0) = C_2$, when $R(\tau \neq 0) \simeq C_2 - \tau^{2H}$ it holds

$$\langle (\Delta X(t))^2 \rangle = 2\,[C_2 - C_2 + \tau^{2H}] = 2\,\tau^{2H} \,. \qquad (35)$$

Definitions in Stochastic Process Theory

Self-similar stochastic process: A stochastic process $X(t)$ whose statistics are the same at different scales of time or space, i.e. $X(a\,t)$ and $a^H X(t)$ have equal statistical moments

$$\langle X^n(a\,t)\rangle \simeq (a\,t)^{nH} = a^{nH} t^{nH} \,, \tag{36}$$

$$\langle [a^H X(t)]^n \rangle \simeq a^{nH} \langle X^n(t) \rangle = a^{nH} t^{nH} \,. \tag{37}$$

Hurst exponent H**:** The half exponent of the power law governing the rate of changes of a random function by

$$\langle [X(t) - X(0)]^2 \rangle \simeq t^{2H} \,. \tag{38}$$

H-sssi Stochastic Processes

H-sssi: Hurst self-similar with stationary increments processes

Example: The fractional Brownian motion, which is a continuos-time Gaussian process $G_{2H}(t)$ without independent increments and the following correlation function

$$\langle G_{2H}(t) G_{2H}(s) \rangle = \frac{1}{2} \left[|t|^{2H} + |s|^{2H} - |t-s|^{2H} \right] . \qquad (39)$$

Product of Random Variables

Let Z_1 and Z_2 be two real independent random variables whose PDFs are $p_1(z_1)$ and $p_2(z_2)$, respectively, with $z_1 \in R$ and $z_2 \in R^+$. Then, the joint PDF is $p(z_1, z_2) = p_1(z_1)p_2(z_2)$.

Let $Z = Z_1 \, Z_2^\gamma$ so that $z = z_1 z_2^\gamma$, then, carrying out the variable transformations $z_1 = z/\lambda^\gamma$ and $z_2 = \lambda$, it follows that $p(z, \lambda) \, dz \, d\lambda = p_1(z/\lambda^\gamma) p_2(\lambda) \, J \, dz \, d\lambda$, where $J = 1/\lambda^\gamma$ is the Jacobian of the transformation.

Integrating in $d\lambda$, the PDF of Z emerges to be

$$p(z) = \int_0^\infty p_1 \left(\frac{z}{\lambda^\gamma} \right) p_2(\lambda) \, \frac{d\lambda}{\lambda^\gamma} \, . \tag{40}$$

Product of Random Variables

Hence by applying the changes of variable $z = xt^{-\gamma\Omega}$ and $\lambda = \tau t^{-\Omega}$, integral formula (40) becomes

$$t^{-\gamma\Omega} p\left(\frac{x}{t^{\gamma\Omega}}\right) = \int_0^\infty \tau^{-\gamma} p_1\left(\frac{x}{\tau^\gamma}\right) t^{-\Omega} p_2\left(\frac{\tau}{t^\Omega}\right) d\tau \,. \qquad (41)$$

By setting

$$p_1 \equiv G \,, \quad p_2 \equiv K^{-\alpha/2}_{\alpha/2,\beta} \,, \qquad (42)$$

$$\gamma = 1/2 \,, \quad \Omega = 2\beta/\alpha \,, \qquad (43)$$

formula (41) turns out to be identical to (26), i.e.

$$p(x;t) = \int_0^\infty G(x;\tau)\, K^{-\alpha/2}_{\alpha/2,\beta}(\tau;t)\, d\tau \,,$$

hence

$$p(x;t) \equiv K^0_{\alpha,\beta}(x;t) \,. \qquad (44)$$

Product of Random Variables

In terms of random variables it follows that

$$Z = X\, t^{-\beta/\alpha} \quad \text{and} \quad Z = Z_1 Z_2^{1/2}\,, \tag{45}$$

hence it holds

$$X = Z\, t^{\beta/\alpha} = Z_1\, t^{\beta/\alpha}\, Z_2^{1/2} = G_{2\beta/\alpha}(t)\, \sqrt{\Lambda_{\alpha/2,\beta}}\,. \tag{46}$$

Since the random variable Z_1 is Gaussian, i.e., $p_1 \equiv G$, the stochastic process $G_{2\beta/\alpha}(t) = Z_1 t^{\beta/\alpha}$ is a standard fBm with Hurst exponent $\beta/\alpha < 1$.

The random variable $\Lambda_{\alpha/2,\beta} = Z_2$ emerges to be distributed according to $p_2 \equiv K_{\alpha/2,\beta}^{-\alpha/2}$.

H-sssi Processes

Following the same constructive approach adopted by Mura (PhD 2008) to built up the *generalized grey Brownian motion* (*Mura, Pagnini J. Phys. A 2008*), the following class of H-sssi processes is established.

Let $X_{\alpha,\beta}(t)$, $t \geq 0$, be an H-sssi defined as

$$X_{\alpha,\beta}(t) \stackrel{d}{=} \sqrt{\Lambda_{\alpha/2,\beta}}\; G_{2\beta/\alpha}(t)\,, \quad 0 < \beta \leq 1\,, \quad 0 < \beta < \alpha \leq 2\,, \qquad (47)$$

where $\stackrel{d}{=}$ denotes the equality of the finite-dimensional distribution, the stochastic process $G_{2\beta/\alpha}(t)$ is a standard fBm with Hurst exponent $H = \beta/\alpha < 1$ and $\Lambda_{\alpha/2,\beta}$ is an independent non-negative random variable with PDF $K^{-\alpha/2}_{\alpha/2,\beta}(\lambda)$, $\lambda \geq 0$, then the marginal PDF of $X_{\alpha,\beta}(t)$ is $K^{0}_{\alpha,\beta}(x;t)$.

H-sssi Processes

Tthe finite-dimensional distribution of $X_{\alpha,\beta}(t)$ is obtained from (40) according to

$$f_{\alpha,\beta}(x_1, x_2, \ldots, x_n; \gamma_{\alpha,\beta}) = \frac{(2\pi)^{-\frac{n-1}{2}}}{\sqrt{\det \gamma_{\alpha,\beta}}} \times \int_0^\infty \frac{1}{\lambda^{n/2}} G\left(\frac{z_n}{\lambda^{1/2}}\right) K_{\alpha/2,\beta}^{-\alpha/2}(\lambda)\, d\lambda\,, \qquad (48)$$

where z_n is the n-dimensional particle position vector

$$z_n = \left(\sum_{i,j=1}^{n} x_i\, {\gamma_{\alpha,\beta}}^{-1}(t_i, t_j)\, x_j \right)^{1/2},$$

and $\gamma_{\alpha,\beta}(t_i, t_j)$ is the covariance matrix

$$\gamma_{\alpha,\beta}(t_i, t_j) = \frac{1}{2}(t_i^{2\beta/\alpha} + t_j^{2\beta/\alpha} - |t_i - t_j|^{2\beta/\alpha})\,, \quad i, j = 1, \ldots, n\,.$$

Stochastic Solution of Space-Time Fractional Diffusion

For the one-point case, i.e., $n = 1$, formula (48) reduces to

$$f_{\alpha,\beta}(x;t) = \int_0^\infty \frac{1}{\lambda^{1/2}} G\left(\frac{x\,t^{-\beta/\alpha}}{\lambda^{1/2}}\right) K_{\alpha/2,\beta}^{-\alpha/2}(\lambda)\,d\lambda = K_{\alpha,\beta}^{0}(x\,t^{-\beta/\alpha})\,, \qquad (49)$$

or, after the change of variable $\lambda = \tau\,t^{-2\beta/\alpha}$,

$$\int_0^\infty \frac{1}{\tau^{1/2}} G\left(\frac{x}{\tau^{1/2}}\right) K_{\alpha/2,\beta}^{-\alpha/2}\left(\frac{\tau}{t^{\beta/\alpha}}\right) d\tau = t^{-\beta/\alpha} K_{\alpha,\beta}^{0}\left(\frac{x}{t^{\beta/\alpha}}\right) . \qquad (50)$$

This means that the marginal PDF of the H-sssi process $X_{\alpha,\beta}(t)$ is indeed the solution of the symmetric space-time fractional diffusion equation (1).

Stochastic Process Generation

From (14) it follows that

$$K_{\alpha/2,\beta}^{-\alpha/2}(\xi;t) = \int_0^\infty L_{\alpha/2}^{-\alpha/2}(\xi;\tau)\, M_\beta(\tau;t)\, d\tau\,, \quad 0 < \beta \le 1\,, \quad (51)$$

and by using the self-similarity properties and the changes of variable $\xi = t^{2\beta/\alpha}\lambda$ and $\tau = t^\beta y$ it holds

$$K_{\alpha/2,\beta}^{-\alpha/2}(\lambda) = \int_0^\infty L_{\alpha/2}^{-\alpha/2}\left(\frac{\lambda}{y^{2/\alpha}}\right) M_\beta(y)\, \frac{dy}{y^{2/\alpha}}\,, \quad 0 < \beta \le 1\,. \quad (52)$$

Stochastic Process Generation

Integral (52) suggests to obtain $\Lambda_{\alpha/2,\beta}$ again by means of the product of two independent random variables, i.e.

$$\Lambda_{\alpha/2,\beta} = \Lambda_1 \cdot \Lambda_2^{2/\alpha} = \mathcal{L}_{\alpha/2}^{ext} \cdot \mathcal{M}_\beta^{2/\alpha} \,, \qquad (53)$$

where $\Lambda_1 = \mathcal{L}_{\alpha/2}^{ext}$ and $\Lambda_2 = \mathcal{M}_\beta$ are distributed according to the extremal stable density $L_{\alpha/2}^{-\alpha/2}(\lambda_1)$ and $M_\beta(\lambda_2)$, respectively, so that $\lambda = \lambda_1 \, \lambda_2^{2/\alpha}$.

Stochastic Process Generation

Moreover, from (16) and setting $t = 1$, the random variable $\mathcal{M}_\beta$ can be determined by an extremal stable random variable according to

$$\mathcal{M}_\beta = \left[\mathcal{L}_\beta^{ext}\right]^{-\beta} , \tag{54}$$

so that the random variable $\Lambda_{\alpha/2,\beta}$ is computed by the product

$$\Lambda_{\alpha/2,\beta} = \mathcal{L}_{\alpha/2}^{ext} \cdot \left[\mathcal{L}_\beta^{ext}\right]^{-2\beta/\alpha} . \tag{55}$$

Finally, the desired H-sssi processes are established as follows

$$X_{\alpha,\beta}(t) = \sqrt{\mathcal{L}_{\alpha/2}^{ext}} \cdot \left[\mathcal{L}_\beta^{ext}\right]^{-\beta/\alpha} G_{2\beta/\alpha}(t) . \tag{56}$$

Numerical Generation (by P. Paradisi)

Computer generation of extremal stable random variables of order $0 < \mu < 1$ is obtained by using the method by Chambers, Mallows and Stuck

$$\mathcal{L}_\mu^{ext} = \frac{\sin[\mu(r_1 + \pi/2)]}{(\cos r_1)^{1/\mu}} \left\{ \frac{\cos[r_1 - \mu(r_1 + \pi/2)]}{-\ln r_2} \right\}^{(1-\mu)/\mu} , \quad (57)$$

where r_1 and r_2 are random variables uniformly distributed in $(-\pi/2, \pi/2)$ and $(0, 1)$, respectively.

Chambers, Mallows, Stuck J. Amer. Statist. Assoc. 1976
Weron Statist. Probab. Lett. 1996

Numerical Generation (by P. Paradisi)

The Hosking direct method is applied for generating the fBm $G_{2H}(t)$, $0 < H < 1$. In particular, first the so-called fractional Gaussian noise Y_{2H} is generated over the set of integer numbers with autocorrelation function

$$\langle Y_{2H}(k) Y_{2H}(k+n) \rangle = \frac{1}{2} \left[|n-1|^{2H} - |n|^{2H} + |n+1|^{2H} \right] . \quad (58)$$

Finally, the fBm is then generated as a sum of stationary increments, i.e. $Y_{2H}(n) = G_{2H}(n+1) - G_{2H}(n)$

$$G_{2H}(n+1) = G_{2H}(n) + Y_{2H}(n) . \quad (59)$$

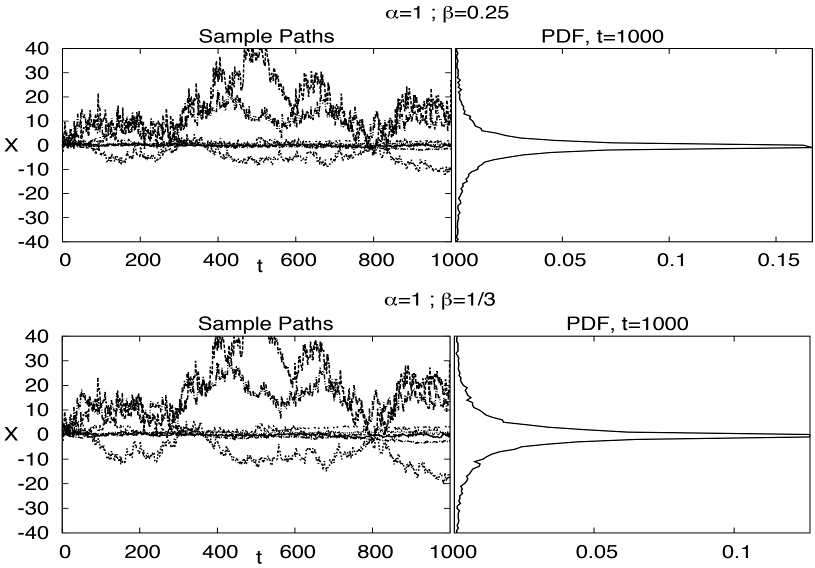
Hosking Water Resour. Res. 1984
Dieker PhD Thesis Univ. of Twente, The Netherlands, 2004

Numerical Set-up (by P. Paradisi)

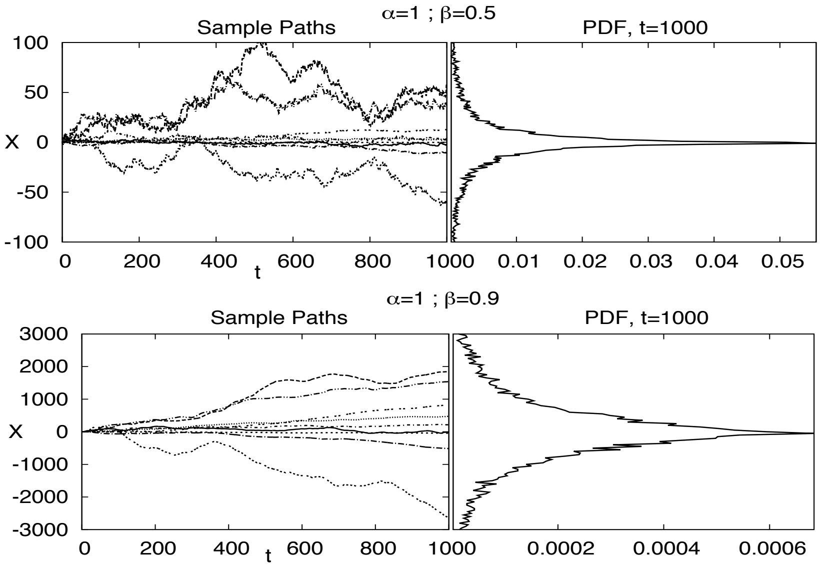
For a given set of parameter values (α,β), 10^4 trajectories are generated and the motion tracked for 10^3 time steps, which is stated equal to 1 following formula (59).

Changing the time scale requires changing the time step, and the associated trajectories can be simply derived without any further numerical simulations by exploiting the self-similar property.

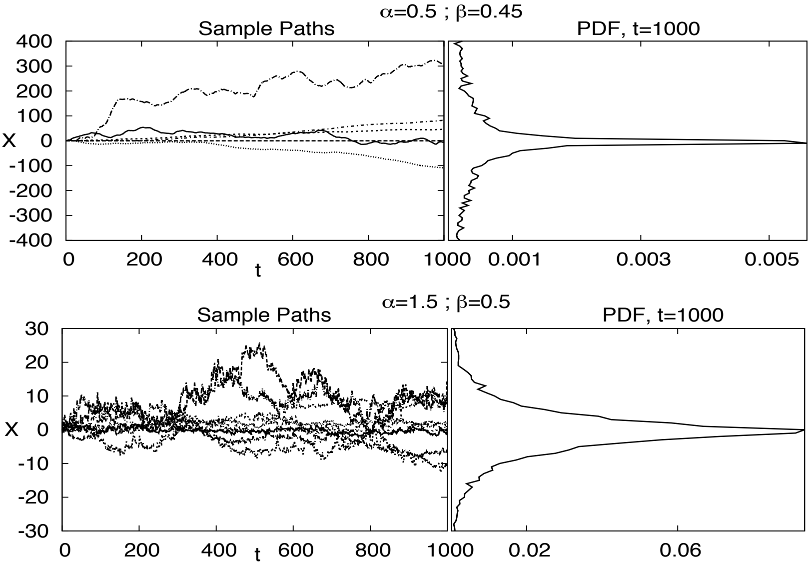
Simulations (by P. Paradisi)



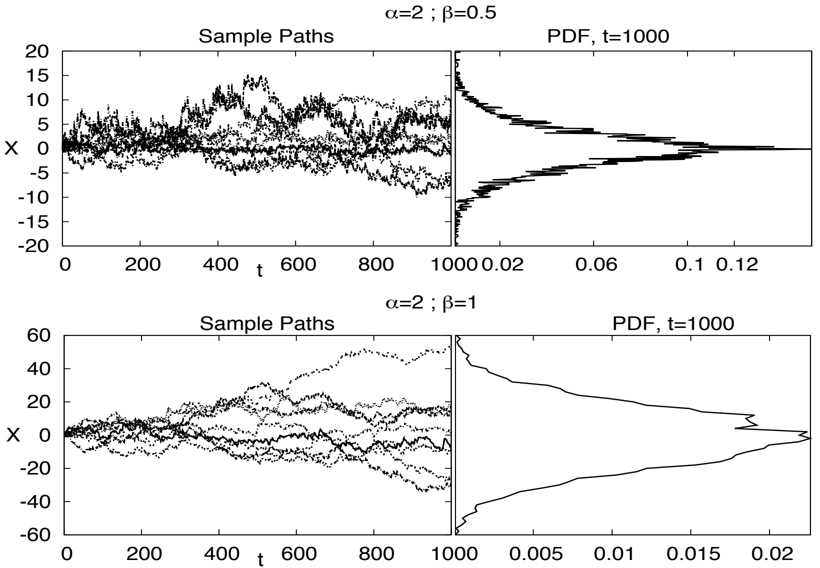
Simulations (by P. Paradisi)



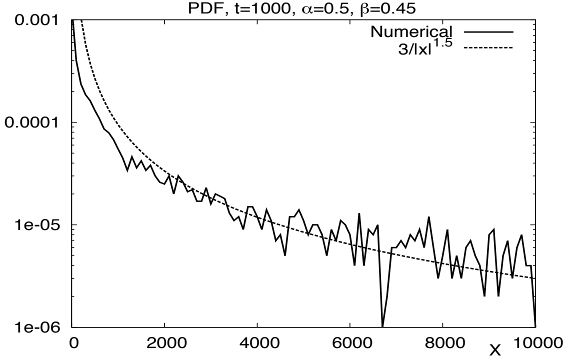
Simulations (by P. Paradisi)



Simulations (by P. Paradisi)



Simulations (by P. Paradisi)



Power-law decay of PDF tail (large x) according to $\sim 1/|x|^{\alpha+1}$.

References

Mainardi F., Luchko Yu., Pagnini G. 2001 The fundamental solution of the space-time fractional diffusion equation. *Fract. Calc. Appl. Anal.* **4**, 153–192.

Mainardi F., Pagnini G., Gorenflo R. 2003 Mellin transform and subordination laws in fractional diffusion processes. *Fract. Calc. Appl. Anal.* **6**, 441-459.

A. Mura 2008 *Non-Markovian Stochastic Processes and Their Applications: From Anomalous Diffusion to Time Series Analysis*. Ph.D. Thesis, University of Bologna.

A. Mura, G. Pagnini 2008 Characterizations and simulations of a class of stochastic processes to model anomalous diffusion. *J. Phys. A: Math. Theor.* **41**, 285003.

G. Pagnini, A. Mura, F. Mainardi 2013 Two-particle anomalous diffusion: Probability density functions and self-similar stochastic processes. *Phil. Trans. R. Soc. A* **371**, 20120154.

Acknowledgements

Thank you for your attention!

This research is supported by the Basque Government through the BERC 2014–2017 program and by the Spanish Ministry of Economy and Competitiveness MINECO: BCAM Severo Ochoa accreditation SEV-2013-0323.

ikerbasque
Basque Foundation for Science

(bcam)
basque center for applied mathematics

EXCELENCIA
SEVERO
OCHOA

GOBIERNO
DE ESPAÑA

MINISTERIO
DE ECONOMÍA
Y COMPETITIVIDAD