From Lévy bases to random vector fields

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This talk is based on a paper

E. Hedevang and J. Schmiegel (2014): A Lévy Based Approach to Random Vector Fields: With a View Towards Turbulence, *Int. J. Nonlinear Sci. Numer. Simul.*

From Lévy bases to random vector fields

We seek to construct random vector fields with a certain set of desirable features.

Computationally tractable on a computer.

Computationally tractable with pen and paper.

Capable of reproducing the features of the problem at hand, e.g., correlation structure (spectrum), distributions (non-Gaussian, one-point, multi-point, increments), volatility/intermittency (energy dissipation, strain), geometric statistics (alignment of vorticity), stationarity/homogeneity/isotropy (or not), etc.

A Gaussian example

Consider a homogeneous Gaussian random vector field,

$$\boldsymbol{v}(\boldsymbol{x}) = \int_{\mathbb{R}^3} F(\boldsymbol{x}-\boldsymbol{y})\boldsymbol{W}(d\boldsymbol{y}),$$

where $F\colon \mathbb{R}^3 \to \mathrm{Mat}_3(\mathbb{R})$ is a deterministic matrix-valued kernel function and W is a homogeneous Gaussian vector-valued white noise,

$$F = \begin{pmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{pmatrix}, \qquad W = \begin{pmatrix} W_1 \\ W_2 \\ W_3 \end{pmatrix}.$$

This model is computationally tractable.

A Gaussian example

$$\boldsymbol{v}(\boldsymbol{x}) = \int_{\mathbb{R}^3} F(\boldsymbol{x}-\boldsymbol{y})\,\boldsymbol{W}(d\boldsymbol{y}) \tag{1}$$

The correlation structure is entirely determined the kernel F,

$$R(\boldsymbol{x}) := \operatorname{cov}(\boldsymbol{v}(\boldsymbol{x}), \boldsymbol{v}(\boldsymbol{0})) = \int_{\mathbb{R}^3} F(\boldsymbol{x}-\boldsymbol{y})F(-\boldsymbol{y})^T\, d\boldsymbol{y}.$$

Equivalently, in Fourier space,

$$\mathscr{F}[R] = (2\pi)^3 \mathscr{F}[F]\mathscr{F}[F]^*.$$

Given a correlation structure R, e.g., isotropic von Kármán, we solve the above equation for F and obtain the representation (1) of the random vector field $\boldsymbol{v}$.

A Gaussian example

$$\boldsymbol{v}(\boldsymbol{x}) = \int_{\mathbb{R}^3} F(\boldsymbol{x}-\boldsymbol{y})\,\boldsymbol{W}(d\boldsymbol{y})$$

In this model, inhomogeneity and anisotropy is possible.

But everything is Gaussian.

If we use the model for turbulence, we get Gaussisan velocity increments, no intermittency of the energy dissipation and incorrect geometric statistics.

Enter Lévy bases and stochastic volatility

$$\boldsymbol{v}(\boldsymbol{x}) = \int_{\mathbb{R}^3} F(\boldsymbol{x}-\boldsymbol{y})\,\sigma(\boldsymbol{y})\,\boldsymbol{L}(d\boldsymbol{y})$$

New integrator, the vector-valued Lévy basis $\boldsymbol{L}$.
New integrand with a stochastic component σ.

A Lévy basis $\boldsymbol{L}$ assigns random variables to subsets of $\mathbb{R}^3$ such that

- $\boldsymbol{L}(A_1),\ldots,\boldsymbol{L}(A_n)$ are independent when $A_1,\ldots,A_n$ are disjoint,
- $\boldsymbol{L}(\bigcup_i A_i) = \sum_i L(A_i)$ when $A_1, A_2, \ldots$ are disjoint,
- $\boldsymbol{L}(A)$ is infinitely divisible.

The first two properties provides a way of constructing the integral.
The infinite divisibility provides a convenient calculus of characteristic functions.

Infinite divisibility

Definition

A random variable X is infinitely divisible if it for any n can be expressed as a sum of n independent random variables,

$$X = X_{n,1} + \cdots + X_{n,n}.$$

Theorem

For an infinitely divisible $\mathbb{R}^3$-valued random variable X, its characteristic function ϕ_X can be written as a certain exponential,

$$\begin{aligned}\phi_X(\boldsymbol{t}) &= \mathbb{E}[\exp(i\langle \boldsymbol{t}, X\rangle)] \\ &= \exp\left(i\langle \boldsymbol{a}, \boldsymbol{t}\rangle - \frac{1}{2}\langle \boldsymbol{t}, B\boldsymbol{t}\rangle + \int_{\mathbb{R}^3}\left(e^{i\langle \boldsymbol{s}, \boldsymbol{t}\rangle} - 1 - i\langle \boldsymbol{s}, \boldsymbol{t}\rangle \mathbb{1}_{\|\boldsymbol{s}\|\le 1}\right) c(d\boldsymbol{s})\right),\end{aligned}$$

where $\boldsymbol{a} \in \mathbb{R}^3$, $B \in \mathrm{Mat}_3^+$ and c is a Lévy measure on $\mathbb{R}^3$.

Infinite divisibility

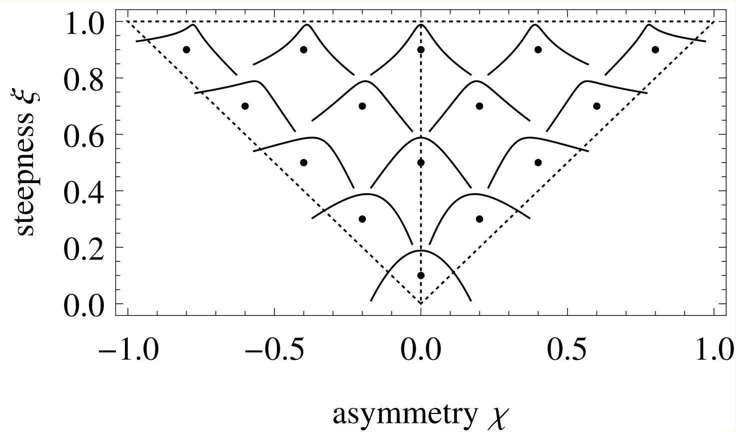
Many distributions are infinitely divisible.

- Normal
- Log-normal
- Normal inverse Gaussian
- Generalised hyperbolic
- Inverse Gaussian
- Generalised Inv. Gaussian
- α-stable
- Tempered stable
- Poisson
- Compound Poisson
- Geometric
- Negative Binomial
- Student's t
- Pareto
- Gumbel
- F
- Weibull
- Logistic
- Half-Caucy
- Self-decomposable
- Generalised Gamma conv.

Some are not: Uniform, Beta.

The normal inverse Gaussian distribution

Many possible shapes with normal inverse Gaussian distributions.



Calculus of characteristic functions

For the model

$$\boldsymbol{v}(\boldsymbol{x}) = \int_{\mathbb{R}^3} F(\boldsymbol{x}-\boldsymbol{y})\,\sigma(\boldsymbol{y})\,\boldsymbol{L}(d\boldsymbol{y})$$

the characteristic function $\phi_{\boldsymbol{v}(\boldsymbol{x})}$ of $\boldsymbol{v}(\boldsymbol{x})$ becomes

$$\phi_{\boldsymbol{v}(\boldsymbol{x})}(\boldsymbol{t}) = \mathbb{E}\left[\exp\left(\int_{\mathbb{R}^3} \log\phi_{\boldsymbol{L}'(\boldsymbol{y})}\left((F(\boldsymbol{x}-\boldsymbol{y})\sigma(\boldsymbol{y}))^T\boldsymbol{t}\right)d\boldsymbol{y}\right)\right],$$

where $\boldsymbol{L}'(\boldsymbol{y})$ is an infinitely divisible random variable.

Think of $\boldsymbol{L}'(\boldsymbol{y})$ as $\boldsymbol{L}(d\boldsymbol{y})$.

If $\sigma = 1$, then

$$\log\phi_{\boldsymbol{v}(\boldsymbol{x})}(\boldsymbol{t}) = \int_{\mathbb{R}^3} \log\phi_{\boldsymbol{L}'(\boldsymbol{y})}(F(\boldsymbol{x}-\boldsymbol{y})^T\boldsymbol{t})\,d\boldsymbol{y}.$$

Building models is not more difficult

Under mild conditions (homogeneous $\boldsymbol{L}$ and σ, $\mathbb{E}[\boldsymbol{L}'] = 0$, $\mathrm{cov}(\boldsymbol{L}') = I$ and $\mathbb{E}[\sigma^2] = 1$), the covariance tensor R becomes

$$R(\boldsymbol{x}) = \mathrm{cov}(\boldsymbol{v}(\boldsymbol{x}), \boldsymbol{v}(\boldsymbol{0})) = \int_{\mathbb{R}^3} F(\boldsymbol{x} - \boldsymbol{y}) F(-\boldsymbol{y})^T \, d\boldsymbol{y},$$

and the spectral tensor $\mathscr{F}[R]$ becomes

$$\mathscr{F}[R] = (2\pi)^3 \mathscr{F}[F] \mathscr{F}[F]^*.$$

Thus, finding a kernel F that reproduces a given covariance tensor R is just as easy (or difficult) as in the purely Gaussian case with no volatility!

Case study: Isotropic incompressible turbulence

The spectral tensor is given in terms of the energy spectrum S,

$$\mathscr{F}[R_{jk}](\boldsymbol{y}) = \frac{S(\|\boldsymbol{y}\|)}{4\pi\|\boldsymbol{y}\|^2}(\delta_{jk} - \hat{y}_j\hat{y}_k)$$

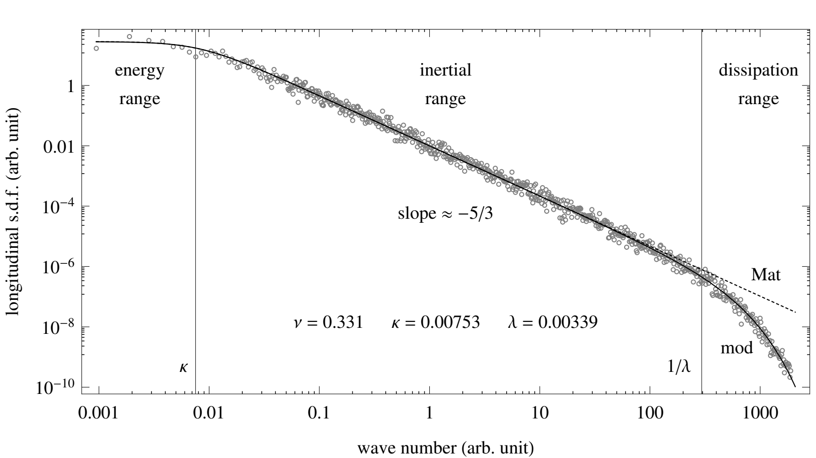
where $\hat{\boldsymbol{y}} = \boldsymbol{y}/\|\boldsymbol{y}\|$.

One can then show that two possible choices of F are the following.

$$F_{jk}^{\text{odd}}(\boldsymbol{x}) = f_1(\|\boldsymbol{x}\|)\epsilon_{jkl}\hat{x}_l,$$
$$F_{jk}^{\text{even}}(\boldsymbol{x}) = f_2(\|\boldsymbol{x}\|)\hat{x}_j\hat{x}_k + f_0(\|\boldsymbol{x}\|)\delta_j k,$$

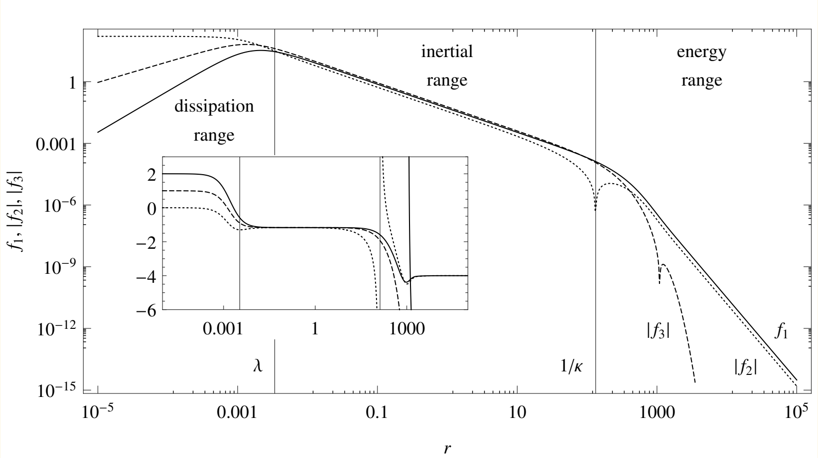
where the functions $f_0, f_1, f_2 : \mathbb{R} \to \mathbb{R}$ are given in terms of certains one-dimensional integrals involving the energy spectrum and the spherical Bessel functions.

Case study: Isotropic incompressible turbulence

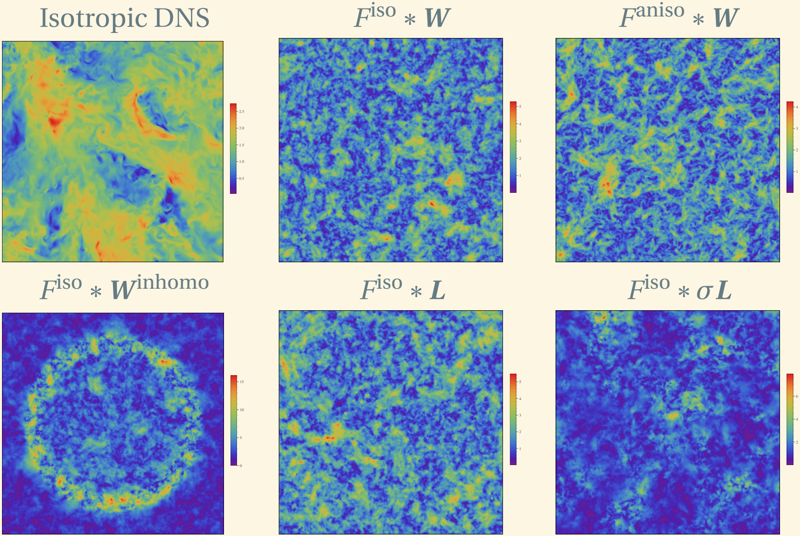


$$S(z) = z^3 \frac{\partial}{\partial z}\left(\frac{1}{z}\frac{\partial}{\partial z}\mathscr{F}[\rho_1](z)\right)$$

Case study: Isotropic incompressible turbulence



Case study: Three-dimensional turbulence



Case study: Non-Gaussian Lévy basis

Consider a simple one-dimensional model with no volatility,

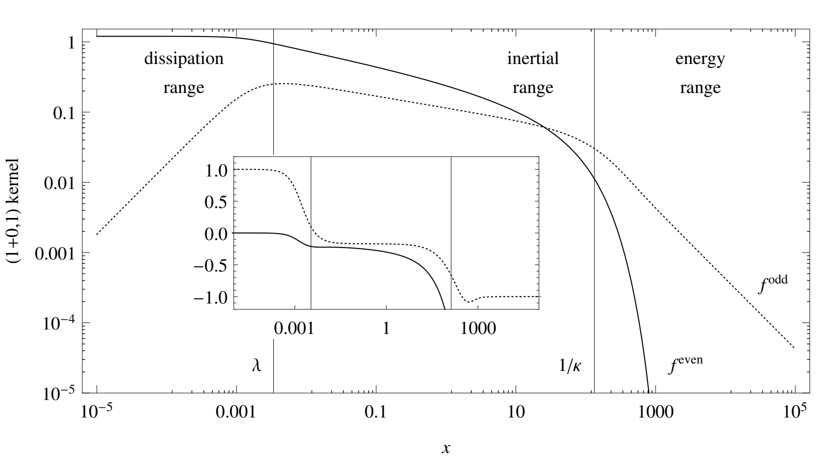
$$v(x) = \int_{\mathbb{R}} f(x-y)\,L(dy),$$

where everything is scalar-valued.

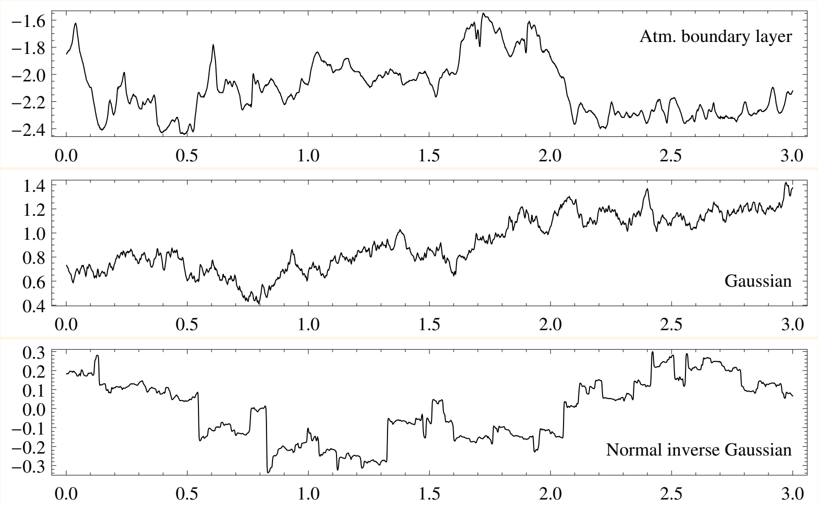
From the spectral density function $\mathscr{F}[\rho_1]$, even and odd kernels may be derived,

$$f^{\text{odd}}(x) = \sqrt{2/\pi}\,\sin\left[\mathscr{F}[\rho_1]^{1/2}\right](x),$$
$$f^{\text{even}}(x) = \sqrt{2/\pi}\,\cos\left[\mathscr{F}[\rho_1]^{1/2}\right](x).$$

Case study: Non-Gaussian Lévy basis



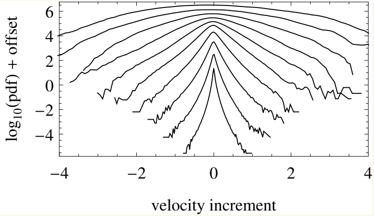
Case study: Non-Gaussian Lévy basis



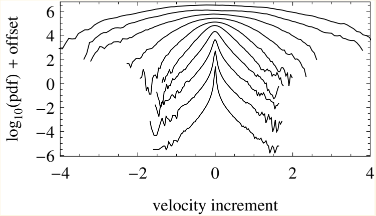
Case study: Non-Gaussian Lévy basis

Velocity increments $v(x+\ell) - v(x)$ should have heavy-tailed and slightly skewed distributions, approaching normal in the limit of large lag ℓ.

atmospheric boundary layer



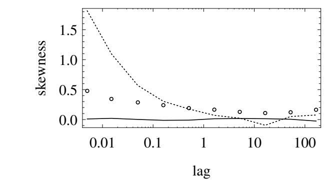
simulation using odd kernel



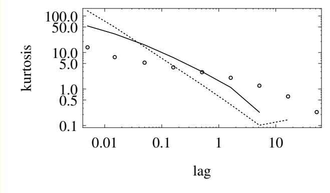
Case study: Non-Gaussian Lévy basis

Without volatility (σ), the correct development of increment distributions and two-point correlators of the energy dissipation is not reproduced. Intermittency is provided by σ.

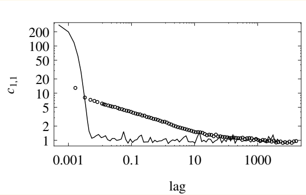
skewness of increments



kurtosis of increments



$c_{1,1}$ of energy dissipation



Summary

For a model like

$$\boldsymbol{v}(\boldsymbol{x}) = \int_{\mathbb{R}^3} F(\boldsymbol{x}-\boldsymbol{y})\,\sigma(\boldsymbol{y})\,\boldsymbol{L}(d\boldsymbol{y}),$$

- F determines the correlation structure
- σ provides volatility and non-Gaussianness
- $\boldsymbol{L}$ can provide non-Gaussianness
- "poor man's" non-Gaussian turbulence is possible with $\sigma = 1$
- anisotropy is possible through F and $\boldsymbol{L}$
- inhomogeneity is possible through σ and $\boldsymbol{L}$

Still remaining to be reproduced: the swirls. Recent results by *Chevillard, Robert and Vargas (2010, Europhys. Lett. 89)* suggest that a matrix-valued σ closely related to the exponential of the strain tensor may hold the key to the swirls.

Outlook

Current ongoing work is to implement simulators of

$$\boldsymbol{v}(\boldsymbol{x}) = \int_{\mathbb{R}^3} F(\boldsymbol{x}-\boldsymbol{y})\,\sigma(\boldsymbol{y})\,\boldsymbol{L}(d\boldsymbol{y})$$

in the following settings

- isotropic, incompressible, periodic boundary conditions: to the used in connection with DNS of the Navier-Stokes equations
- anisotropic, incompressible, atmospheric boundary layer turbulence: to be used in connection with wind energy
- particle transport...

Spatio-temporal models

$$\boldsymbol{v}(\boldsymbol{x},t) = \int_{A(\boldsymbol{x},t)} F(\boldsymbol{x}-\boldsymbol{y},t-s)\,\sigma(\boldsymbol{y},s)\,\boldsymbol{L}(d\boldsymbol{y}\,ds)$$