GERBER-SHIU TYPE FORMULAS FOR RANDOM WALKS

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RANDOM WALK $(X_n)_{n\geq 0}$,

$$X_n = x - \sum_{i=1}^{n} Y_i$$

with the Y_i iid.

Assume that $x \geq 0$ and consider the *exit time*

$$\tau = \inf \{n \geq 1 : X_n < 0\}$$

with $\inf \varnothing = \infty$ and the *value at exit* X_τ (defined on $(\tau < \infty)$ only).

PROBLEM 1: determining the joint distribution of (τ, X_τ) (MJ paper in festschrift);

PROBLEM 2: determining the joint distribution of $(\tau, X_{\tau-1}, X_\tau)$ (this talk).

Both problems discussed under the following assumptions on the distribution F of the Y_i: write

$$F = pF_+ + qF_-$$

with F_+, F_- the restrictions of F to $]0,\infty[$, $]-\infty,0]$ and $q = 1 - p$; assume $0 < p \le 1$ and that F_+ is a *linear combination of exponentials,*

$$F_+(\mathrm{d}x) = \sum_{i=1}^{m} \alpha_i \mu_i e^{-\mu_i x}\, \mathrm{d}x \quad (x > 0)$$

with $\sum \alpha_j = 1$ and, e.g, $0 < \mu_1 < \cdots < \mu_m$; of course it is also required that the density is ≥ 0.

Laplace transforms

$$L_+(\theta) = \int e^{-\theta x}\, F_+(\mathrm{d}x) = \sum_{i=1}^{m} \alpha_i \frac{\mu_i}{\mu_i + \theta} \quad (\mathrm{Re}\,\theta \ge 0),$$

$$L_-(\theta) = \int e^{-\theta x}\, F_-(\mathrm{d}x) \quad (\mathrm{Re}\,\theta \le 0).$$

Note. L_+ has analytic extension $\overline{L}_+$ to $\mathbf{C}$ except for the poles $-\mu_j$.

Combined 'Laplace transform'

$$\widetilde{L}\,(\theta) = p\overline{L}_+(\theta) + qL_-\,(\theta)$$

used only if $\mathrm{Re}\,\theta \leq 0$.

METHOD (partial eigenfunctions). $(Z_n)_{n\geq 0}$ Markov chain on $(E, \boldsymbol{E})$, $A \in \boldsymbol{E}$, $Z_0 \equiv z \in A^c$,

$$\tau_A = \inf\{n \geq 0 : Z_n \in A\}\,.$$

Let $0 < t < 1$, let $g : E \to \mathbf{R}$ be bounded and measurable and define

$$\psi_t(z) = \mathbb{E}_z t^\tau g(Z_\tau) \quad (z \in E)$$

and let P denote the transition operator

$$Ph(z) = \mathbb{E}_z h(Z_1).$$

If $\phi_t : E \to \mathbf{R}$ *is bounded and measurable and satisfies*

$$P\phi_t(z) = \begin{cases} t^{-1}\phi_t(z) & (z \in A^c)\,, \\ g(z) & (z \in A)\,, \end{cases}$$

then $\phi_t \equiv \psi_t$.

PROBLEM 1. For the random walk the transition operator is

$$Ph(x) = \int h(x-y)\,F(\mathrm{d}y).$$

Find $\mathbb{E}_x t^\tau e^{\zeta X_\tau}$ for $x \geq 0$, $0 < t < 1$, $\zeta \geq 0$ by finding partial eigenfunction ϕ_t such that

$$P\phi_t(x) = \begin{cases} t^{-1}\phi_t(x) & (x \geq 0)\,, \\ e^{\zeta x} & (x < 0)\,. \end{cases}$$

SOLUTION. The Cramér-Lundberg equation

$$\widetilde{L}(\gamma) = t^{-1}$$

has precisely m solutions γ with $\mathrm{Re}\,\gamma < 0$ (counted with multiplicity). If these t-dependent solutions $\gamma_1^\circ, \ldots, \gamma_m^\circ$ are distinct,

$$\mathbb{E}_x t^\tau e^{\zeta X_\tau} = \sum_{j=1}^{m} c_j^\circ(t,\zeta) e^{\gamma_j^\circ x} \quad (x \geq 0)$$

with the $c_j^\circ(t,\zeta)$ the solutions to the linear system

$$\sum_{j=1}^{m} c_j^\circ(t,\zeta) \frac{1}{\mu_l + \gamma_j^\circ} = \frac{1}{\mu_l + \zeta} \quad (1 \leq l \leq m)\,.$$

Furthermore, $\mathbb{E}_x\left[e^{\zeta X_\tau}; \tau < \infty\right] = \lim_{t\uparrow 1} \mathbb{E}_x t^\tau e^{\zeta X_\tau}$ is given by

$$\mathbb{E}_x\left[e^{\zeta X_\tau}; \tau < \infty\right] = \sum_{j=1}^{m} c_j^\circ(1,\zeta) e^{\gamma_j^\circ x} \quad (x \geq 0)$$

where $\gamma_1^\circ, \ldots, \gamma_m^\circ$ are the solutions to $\widetilde{L}(\gamma) = 1$ determined as follows:

(i) if $\mathbb{E}Y_1 < 0$ ($\mathbb{P}_x(\tau < \infty) < 1$), the precisely m solutions with $\operatorname{Re}\gamma < 0$;

(ii) if $\mathbb{E}Y_1 \geq 0$ ($\mathbb{P}_x(\tau < \infty) = 1$), the precisely $m - 1$ solutions with $\operatorname{Re}\gamma < 0$ and the solution 0.

PROBLEM 2. Consider the Markov chain $(Z_n)_{n\geq 0}$ given by

$$Z_n = (X_{n-1}, X_n)$$

with transition operator

$$Ph\,(w,x) = \int h\,(x, x-y)\ F(\mathrm{d}y).$$

Take $A = \{(w,x) : w < 0 \text{ or } x < 0\}$, then $\tau_A = \tau$ when $Z_0 \equiv (w,x)$ with $w \geq 0$, $x \geq 0$.

Find $\mathbb{E}_x t^\tau e^{-\theta X_{\tau-1}+\zeta X_\tau}$ for $x \geq 0$, $0 < t \leq 1$, $\theta \geq 0$, $\zeta \geq 0$ by finding partial eigenfunction ϕ_t such that

$$P\phi_t(w,x) = \begin{cases} t^{-1}\phi_t(w,x) & (w \geq 0, x \geq 0)\,, \\ e^{-\theta w+\zeta x} & (w \geq 0, x < 0)\,. \end{cases}$$

SOLUTION (found by guessing, guessing again —). Let $\gamma_1^\circ, \ldots, \gamma_m^\circ$ be as above (also for $t = 1$), assume $\theta > 0$, $\theta \neq$ all $\mu_k - \mu_l$ and all $\mu_i + \theta \neq$ from all $-\gamma_j$ (!). Then for $x \geq 0$

$$\mathbb{E}_x t^\tau e^{-\theta X_{\tau-1} + \zeta X_\tau} = \sum_{i=1}^m d_i e^{-(\mu_i+\theta)x} - \sum_{i=1}^m b_i \mathbb{E}_x t^\tau e^{\mu_i X_\tau} \qquad (1)$$

with

$$d_i = d_i\left(t, \theta, \zeta\right) = \frac{p\alpha_i \mu_i / \left(\mu_i + \zeta\right)}{t^{-1} - \widetilde{L}\left(-\left(\mu_i + \theta\right)\right)} \qquad (2)$$

and the $b_i = b_i\left(t, \theta, \zeta\right)$ the solutions to the linear system

$$\sum_{i=1}^m b_i \frac{1}{\mu_i + \mu_l} = \sum_{i=1}^m d_i \frac{1}{\mu_l - \mu_i - \theta} \quad \left(1 \leq l \leq m\right).$$

Note that from PROBLEM 1 we know that

$$\mathbb{E}_x t^{\tau} e^{\mu_i X_\tau} = \sum_{j=1}^{m} c_{ji}^{*} e^{\gamma_j^{\circ} x}$$

with, for each i, the c_{ji}^{*} the solutions to the linear system

$$\sum_{j=1}^{m} c_{ji}^{*} \frac{1}{\mu_l + \gamma_j^{\circ}} = \frac{1}{\mu_l + \mu_i} \quad (1 \leq l \leq m).$$

IMPORTANT REMARK. The denominator on the right of (2) vanishes if $\mu_i + \theta = -\gamma_j^\circ$ and has singularities at the other θ-values excepted above. The singularities for $\theta = 0$ or $\theta = \mu_k - \mu_l$ arises from the contribution

$$p \sum_{k=1}^{m} \alpha_k \frac{\mu_k}{\mu_k - \mu_i - \theta}$$

to $\widetilde{L}\left(-\left(\mu_i + \theta\right)\right)$. In particular $\lim_{\theta \downarrow 0} d_i = 0$ which helps understanding why $\theta \downarrow 0$ in (1) gives the formula from PROBLEM 1 a s it should. Across the other singularities the right hand side of (1) should be continuous in θ, useful to check.

ASYMPTOTICS AS $x \to \infty$. Assume $\mathbb{E}Y_1 \geq 0$ and take $t = 1$. $\gamma_1^\circ = 0$ is the 'largest' solution to the CL-equation; in (1) only the terms with $e^{\gamma_1^\circ x} = 1$ remain as $x \to \infty$:

$$\lim_{x\to\infty} \mathbb{E}_x e^{-\theta X_{\tau-1} + \zeta X_\tau} = -\sum_{i=1}^{m} b_i c_{1i}^*.$$

For $m = 1$,

$$d = \frac{p\mu / (\mu+\zeta)}{1 - \widetilde{L}(-(\mu+\theta))}, \quad b = -\frac{2d\mu}{\theta}, \quad c^* = \frac{1}{2}$$

so

$$\lim_{x\to\infty} \mathbb{E}_x e^{-\theta X_{\tau-1} + \zeta X_\tau} = \frac{p\mu^2 / (\mu+\zeta)}{\theta\left(1 - qL_-(-(\mu+\theta)) + \frac{p\mu}{\theta}\right)}.$$

For $\theta = 0$ gives $X_\tau \sim \exp(\mu)$ (as it should). $\zeta = 0$ or using $X_{\tau-1} \perp X_\tau$ gives

$$\lim_{x\to\infty} \mathbb{E}_x e^{-\theta X_{\tau-1}} = \frac{p\mu}{\theta\left(1 - qL_-(-(\mu+\theta)) + \frac{p\mu}{\theta}\right)}.$$

If F_- is $\exp(\kappa)$ on $]-\infty, 0]$, $L_-(\nu) = \frac{\kappa}{\kappa-\nu}$ for $\nu \leq 0$:

$$\lim_{x \to \infty} \mathbb{E}_x e^{-\theta X_{\tau-1}} = \frac{p\mu}{\theta\left(1 - q\frac{\kappa}{\kappa+\mu+\theta} + \frac{p\mu}{\theta}\right)}.$$

For $p = 1$ of course $X_{\tau-1} \sim \exp(\mu)$. Otherwise $X_{\tau-1} \sim$ linear combination of two exponentials, rates μ and $p(\kappa+\mu)$, weights $p\kappa/(p\kappa - q\mu)$ and $-q\mu/(p\kappa - q\mu)$ provided $p\kappa - q\mu > 0$. ($\mathbb{E}Y_1 \geq 0 \Leftrightarrow p\kappa - q\mu \geq 0$).

CONTINUOUS TIME. $(Z_t)_{t\geq 0}$ cadlag Markov, $Z_0 \equiv z$, use Itô decompositions

$$e^{-\rho\tau_A \wedge t} h(Z_{\tau_A \wedge t}) = h(z) + \int_0^{\tau_A \wedge t} e^{-\rho s} \left(\mathbf{A}h(Z_s) - \rho h(Z_s)\right) \mathrm{d}s + M_t$$

with $\mathbf{A}$ infinitesimal generator, (M_t) martingale. ϕ_ρ partial eigenfunction:

$$\mathbf{A}\phi_\rho(z) = \begin{cases} \rho\phi_t(z) & (z \in A^c), \\ g(z) & (z \in A). \end{cases}$$

Simple analogue of random walk: (X_t) compound Poisson,

$$X_t = x + \beta t - \sum_{n=1}^{N_t} Y_n$$

with the Y_n iid, same type of distribution as before, $\beta \geq 0$.

PROBLEM 1: special case of much more general model, MJ, AAP 2005.

PROBLEM 2: can be done, can of course not use the analogue (X_{t-}, X_t) of (X_{n-1}, X_n), forced to use (X_t, X_t)!