Stochastic modelling of the turbulent velocity field

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Goal:

Modelling the **(3+1)-dimensional** turbulent velocity field as a stochastic process that captures main stylized facts of turbulent data.

Today:

Modelling the **(0+1)-dimensional** turbulent velocity field as a stochastic process that captures main stylized facts of turbulent data

Stylized facts of turbulent data:

- Heavy tailed distributions
- Scaling of structure functions
- Scaling of energy dissipation correlators
- Statistics of the Kolmogorov variable

Heavy tailed distributions

The probability densities of velocity increments $u_s = v_s - v_0$ evolve from heavy tails at small time scales s towards an approximate Gaussian shape at large time scales s.

The evolution of densities across time scales is well approximated within the class of normal inverse Gaussian distributions.

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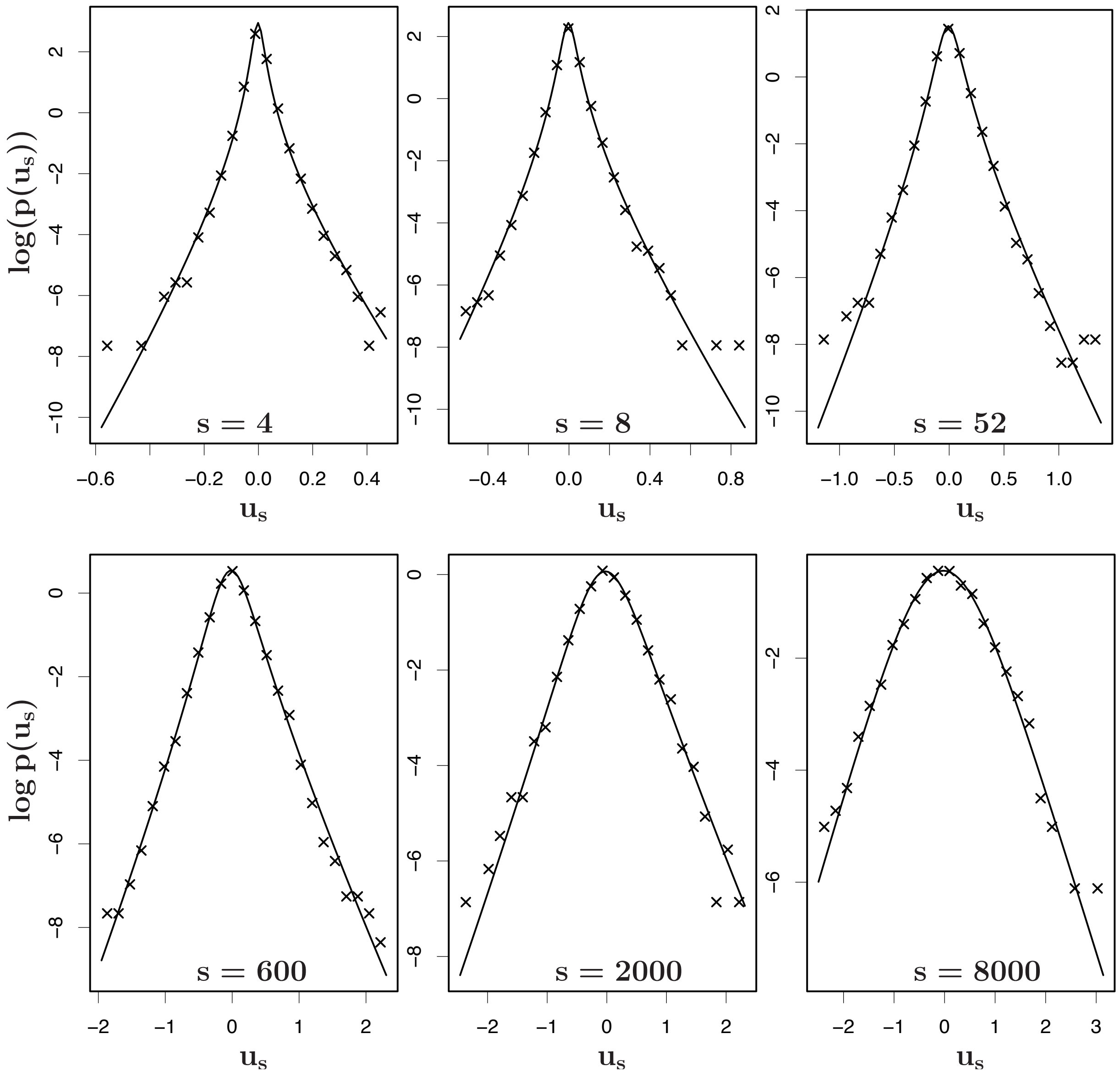
Heavy tailed distributions: atmospheric boundary layer: $R_\lambda = 17000$



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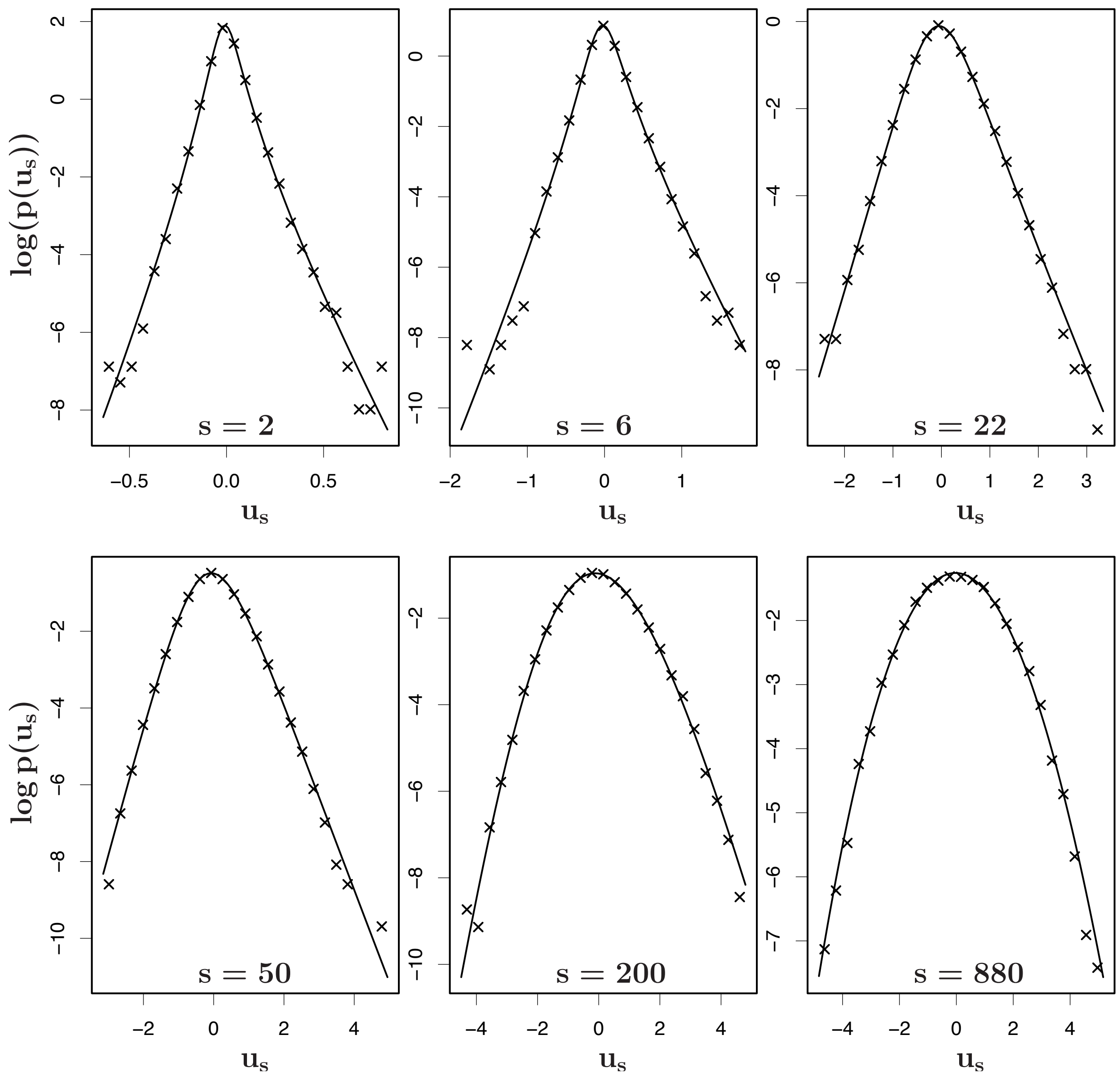
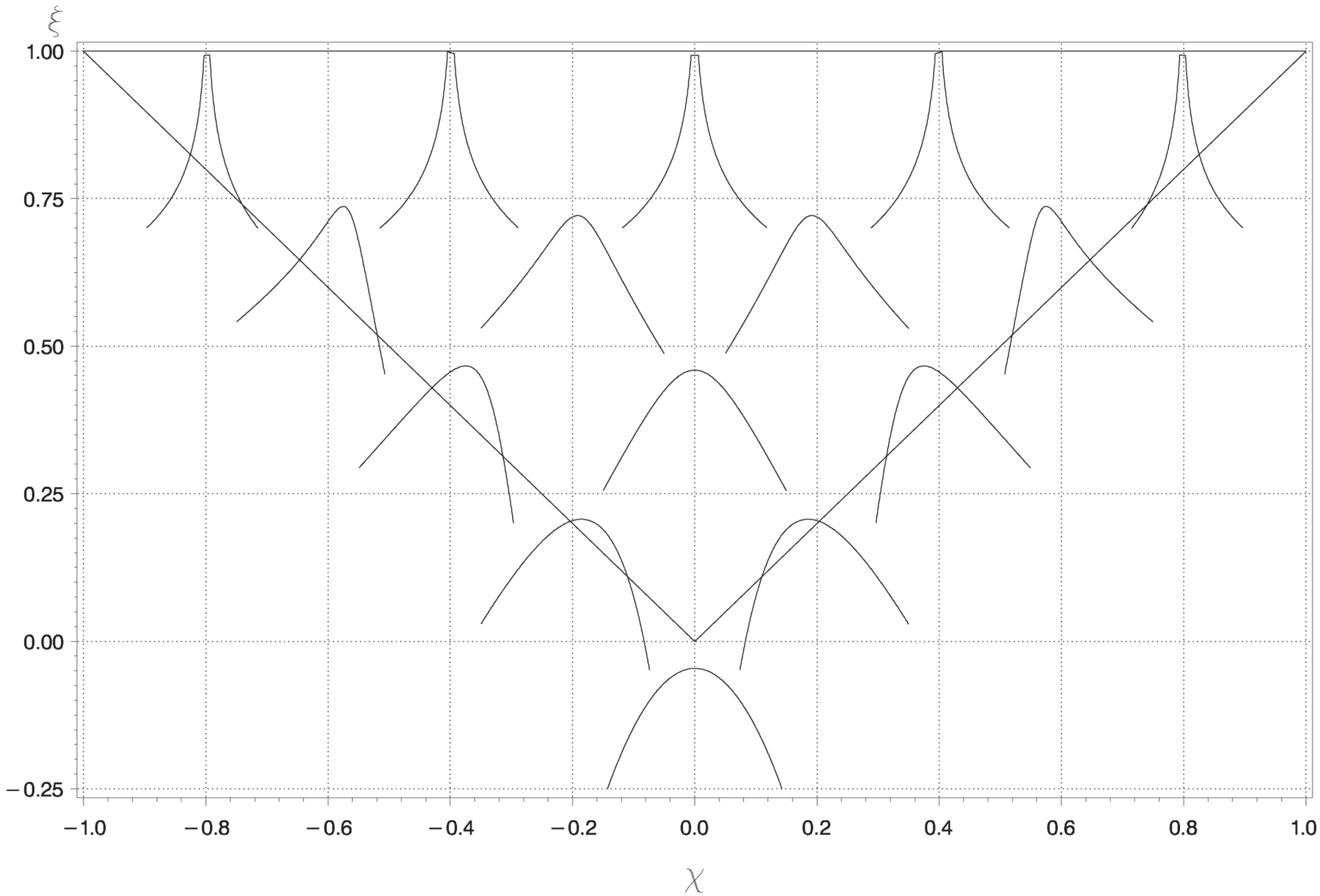
Heavy tailed distributions: jet experiment: $R_\lambda = 190$



NIG shape triangle



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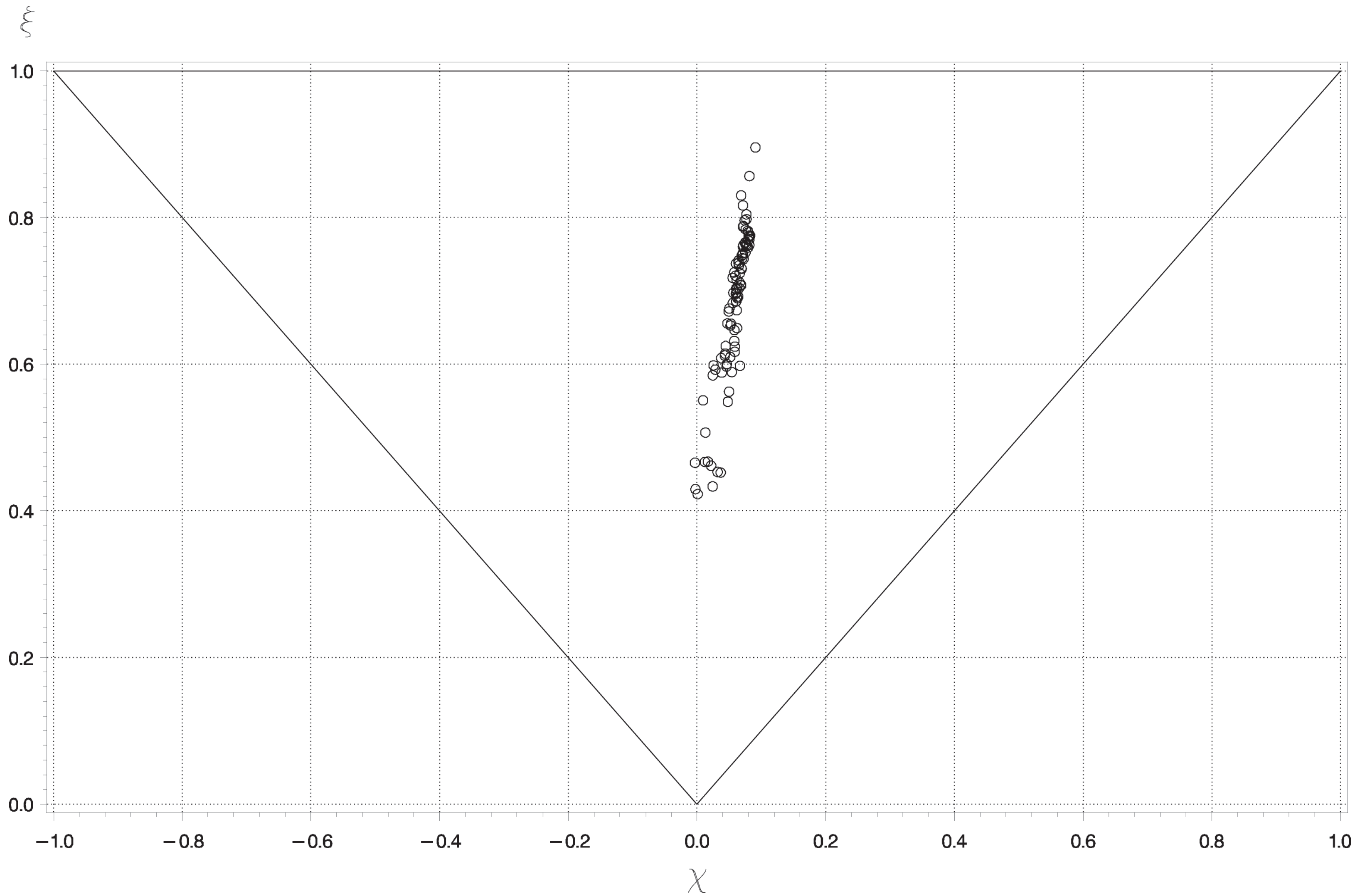

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Scaling of structure functions

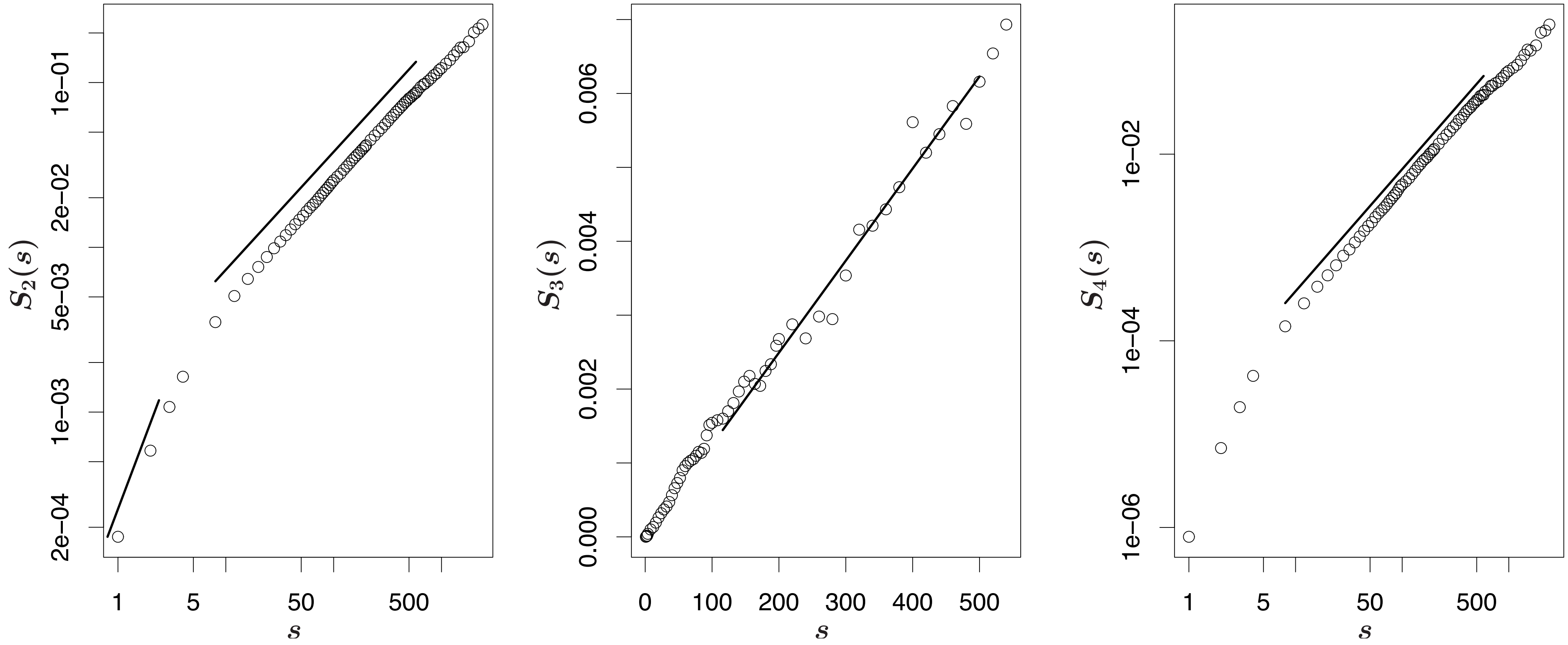
Structure functions of velocity increments of order n

$$S_n(s) = E\left\{u_s^n\right\} = E\left\{\left(v_s - v_0\right)^n\right\} \propto s^{-\tau(n)}$$

show (approximate) scaling behaviour for a range of time scales s (the so-called inertial range) in the limit of large Reynolds numbers R_λ.

Scaling of structure functions

atmospheric boundary layer: $R_\lambda = 17000$



Scaling of energy dissipation correlators

Two-point correlators $c_{n_1,n_2}(s)$ at time scale s and order (n_1, n_2) of the (surrogate) energy dissipation (ν denotes the viscosity)

$$\varepsilon_t(x) = 15\nu \left(\frac{\partial v_t(x)}{\partial t} \right)^2$$

are defined as

$$c_{n_1,n_2}(s) = \frac{E\left\{\varepsilon_t(x)^{n_1} \varepsilon_{t+s}(x)^{n_2}\right\}}{E\left\{\varepsilon_t(x)^{n_1}\right\} E\left\{\varepsilon_{t+s}(x)^{n_2}\right\}}.$$

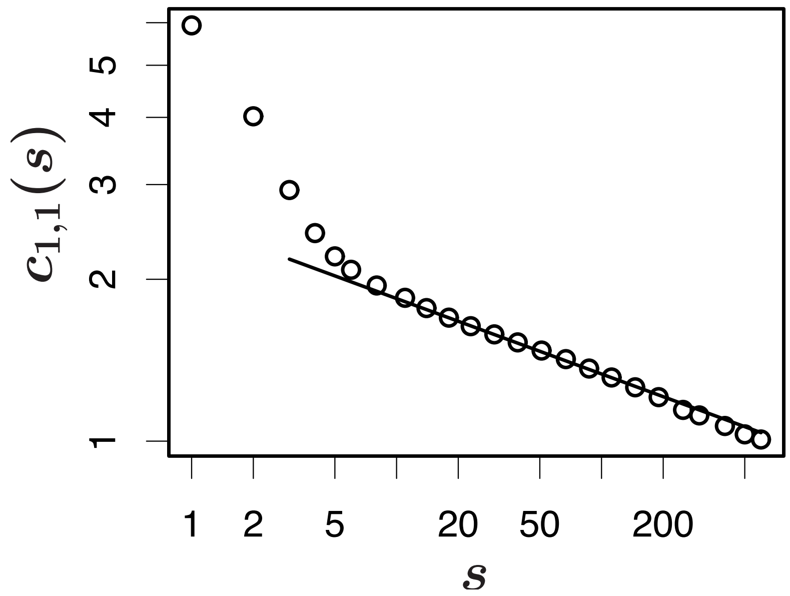
These correlators show (approximate) scaling behaviour

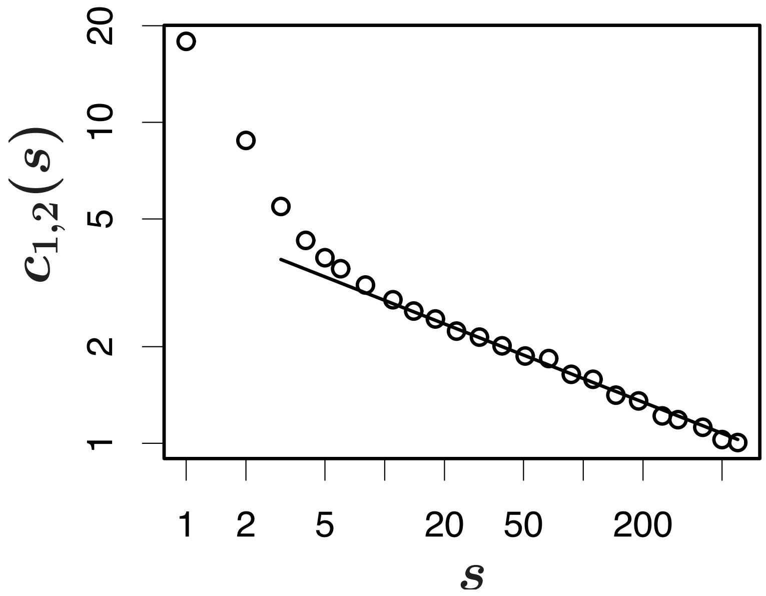
$$c_{n_1,n_2}(s) \propto s^{-\xi(n_1,n_2)}$$

for time scales s within the inertial range.

Scaling of energy dissipation correlators

helium jet experiment: $R_\lambda = 208$





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Statistics of the Kolmogorov variable

The Kolmogorov variable V_t (in the time domain) is defined as

$$V_t = \frac{u_t}{(\bar{\varepsilon}_t)^{1/3}}$$

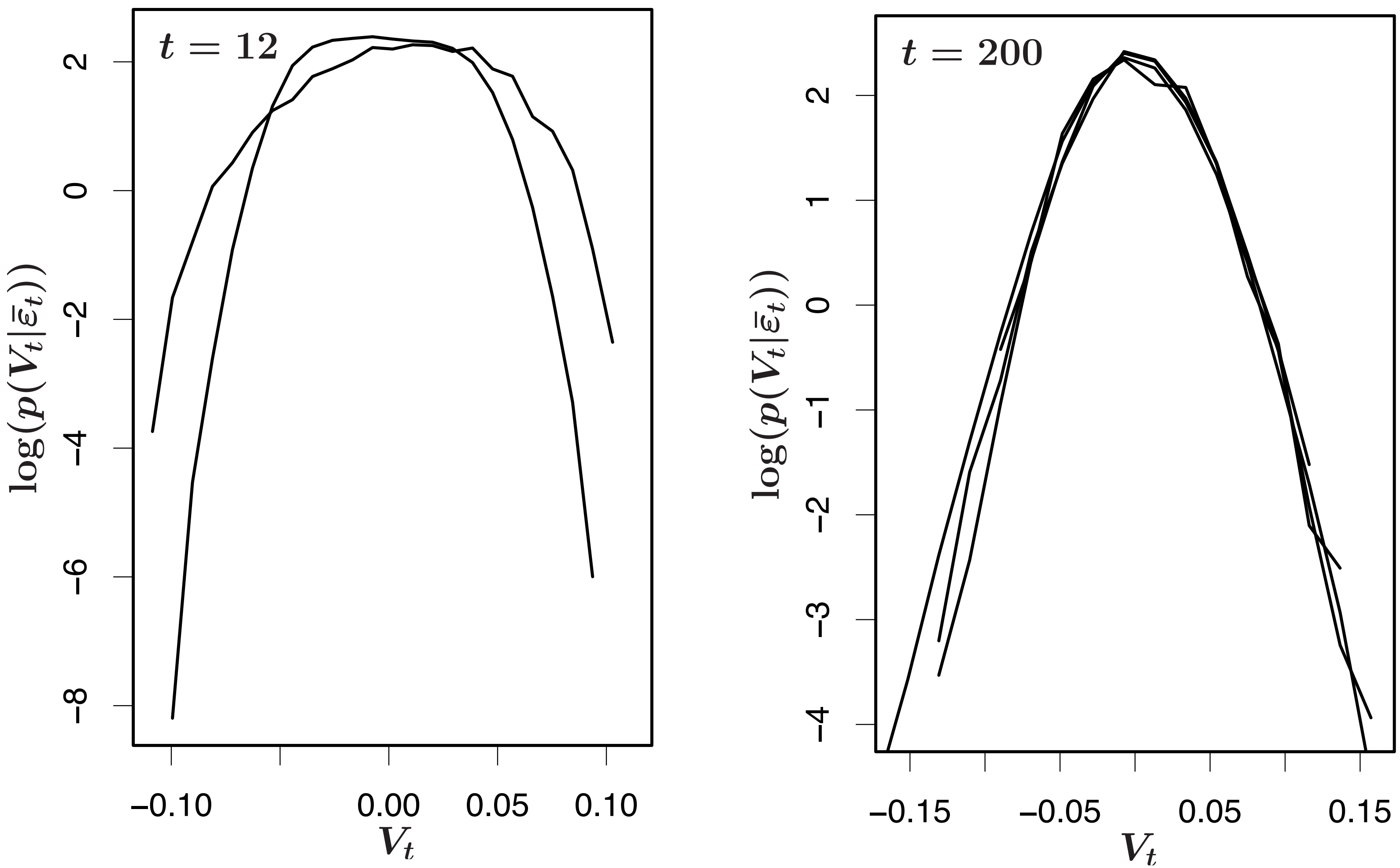
where

$$\bar{\varepsilon}_t = \int_0^t \varepsilon_s ds$$

is the integrated surrogate energy dissipation.
The conditional probability densities $p\left(V_t \middle| \bar{\varepsilon}_t\right)$ do not depend on $\bar{\varepsilon}_t$ for t within the inertial range.

Statistics of the Kolmogorov variable

atmospheric boundary layer: $R_\lambda = 17000$



The (0+1)-dimensional modelling framework

We propose to model the timewise dynamics of the main component of the turbulent velocity field v_t at a fixed position and at time t as a stochastic process of the form

$$v_t = \int_{-\infty}^{t} f(t-s)\sqrt{J_s}\,dB_s + \beta \int_{-\infty}^{t} f(t-s)J_s\,ds$$

where B denotes Brownian motion, f is a deterministic kernel and β is a constant. The stochastic process J is called the intermittency process.

The velocity v_t is not differentiable,

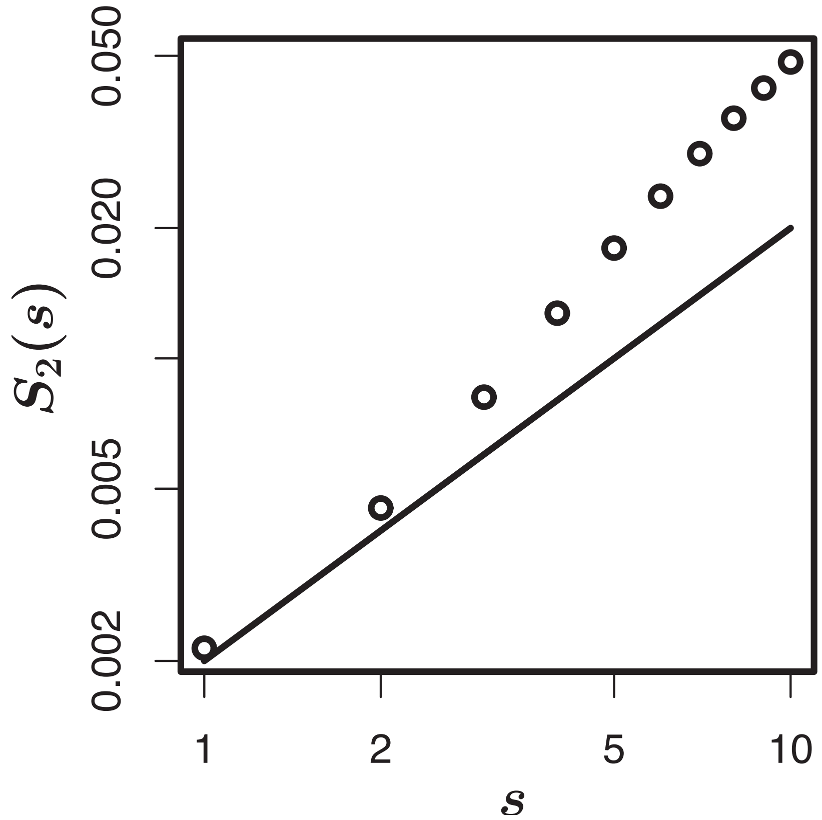
but

$$\frac{(dv_t)^2}{dt} = f(0)^2 J_t$$

and

$$S_2(t) \propto t$$

for small t (small scale diffusion).



helium jet experiment: $R_\lambda = 283$

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Structure functions

$$S_2(t) = E\{J\}F(t) + \beta^2 G(t)$$

where

$$G(t) = \int_{-\infty}^{t}\int_{-\infty}^{t} \tilde{f}(t,s)\tilde{f}(t,s')E\{J_s J_{s'}\}ds\,ds'$$

$$F(t) = \int_{-\infty}^{t} \tilde{f}(t,s)^2\, ds$$

$$\tilde{f}(t,s) = f(t-s) - I_{(-\infty,0]}(s) f(-s)$$

Structure functions

$$S_3(t) = \beta B_1(t) + \beta^3 B_3(t)$$

where

$$B_1(t) = \int_{-\infty}^{t} \int_{-\infty}^{t} \tilde{f}(t,s)^2 \tilde{f}(t,s') E\{J_s J_{s'}\} ds ds'$$

$$B_3(t) = \int_{-\infty}^{t} \int_{-\infty}^{t} \int_{-\infty}^{t} \tilde{f}(t,s) \tilde{f}(t,l) \tilde{f}(t,w) E\{J_s J_l J_w\} ds dl dw$$

Intermittency

$$J_t(\triangle t) = \frac{(v_{t+\triangle t} - v_t)^2}{\triangle t} \propto \varepsilon_t(\triangle t)\triangle t$$

We model the intermittency process J at a fixed position σ as a continuous cascade process

$$J_t = \exp\left\{ \int_{t-T}^{t} \int_{\sigma - r(s-t+T)}^{\sigma + r(s-t+T)} dZ \right\}$$

where Z is a Lévy basis, T denotes the decorrelation time and

$$r(t) = \frac{a}{t+b}$$

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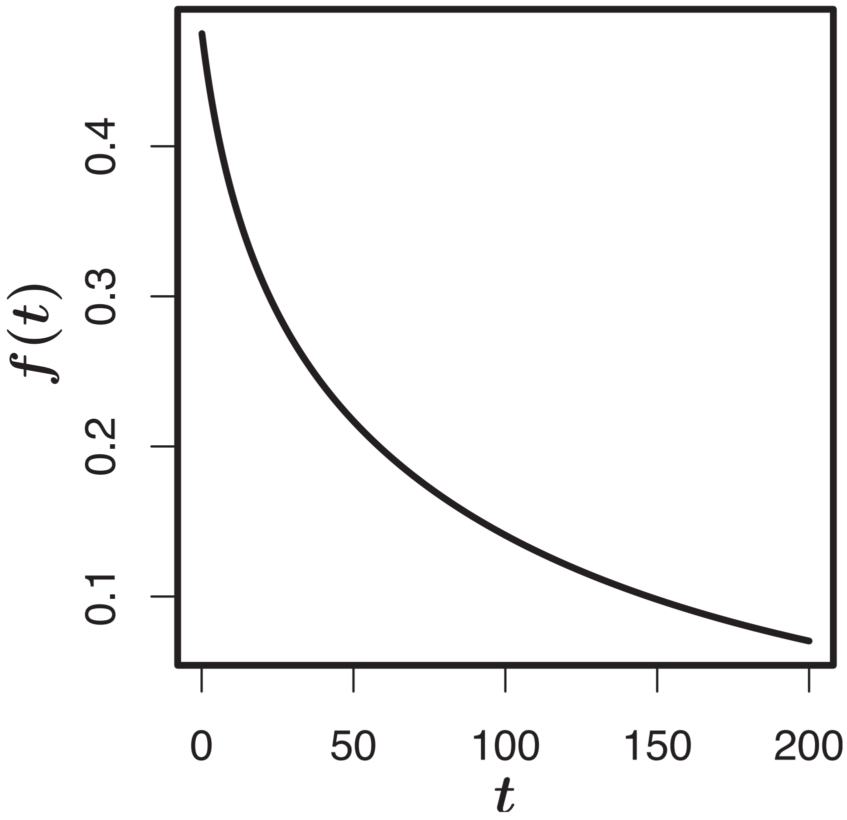
Intermittency: Scaling of correlators

$$c_{n_1,n_2}(s) = \frac{E\left\{J_0^{\,n_1} J_s^{\,n_2}\right\}}{E\left\{J_0^{\,n_1}\right\} E\left\{J_s^{\,n_2}\right\}} \propto (s+b)^{-\tau(n_1,n_2)}$$

Weight function

$$f(t) = (t + t_0)^{-\nu} e^{-\lambda(t+t_0)}$$



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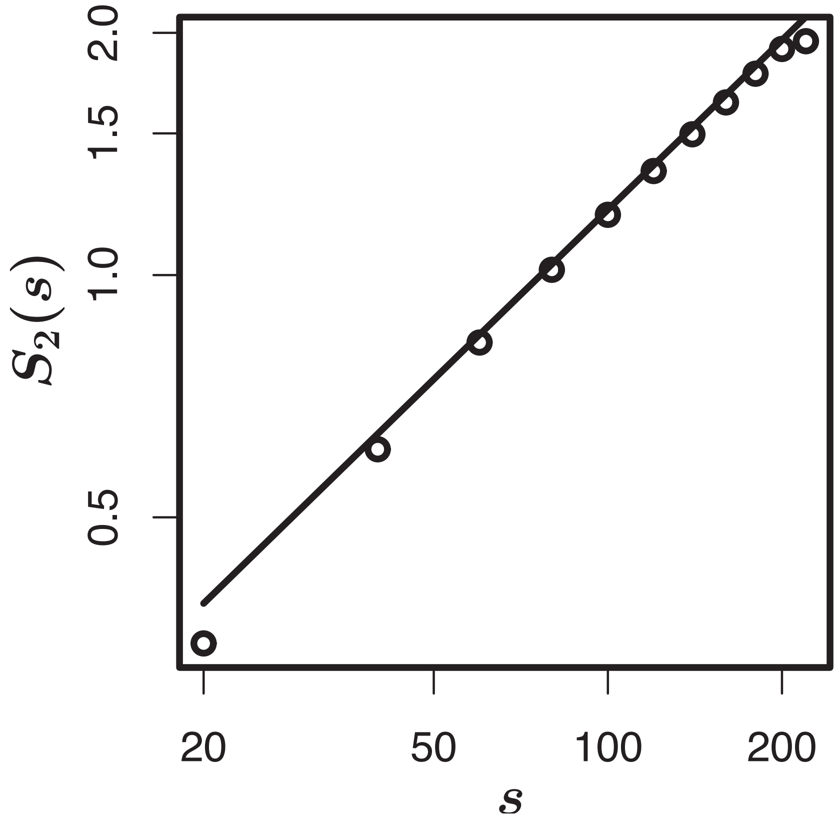
Scaling of second order structure function

$$f(t) = (t + t_0)^{-\nu} e^{-\lambda(t+t_0)}$$

$$S_2(t) = E\{J\}F(t) + \beta^2 G(t)$$

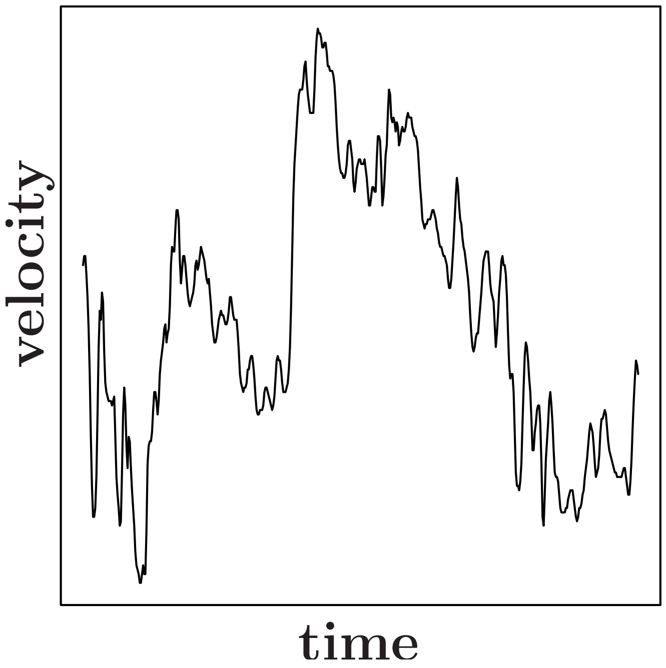


For the simulation of the model we choose a normal Lévy basis for the intermittency process J and a weight function f of the form

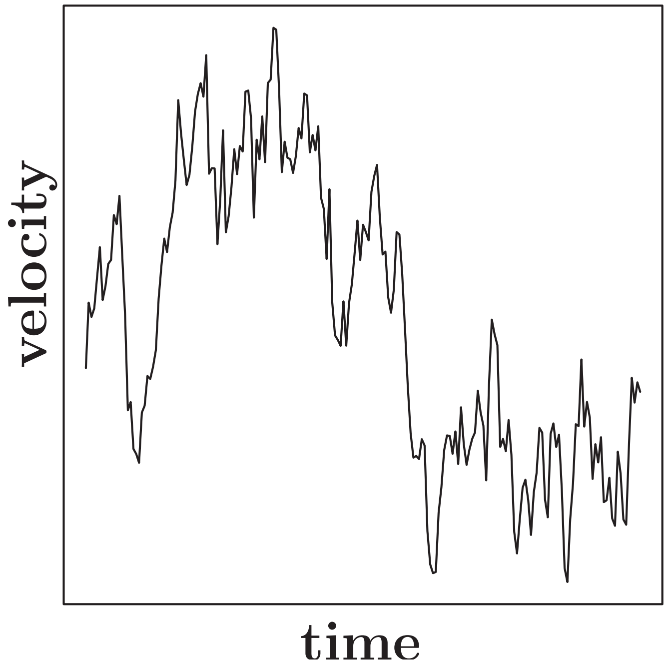
$$f(t) = \begin{cases} (t + t_0)^{-\nu} e^{-\lambda(t+t_0)}, & 0 \leq t \leq T \\ 0, & \text{otherwise} \end{cases}$$

Time series: velocity

Data



Simulation



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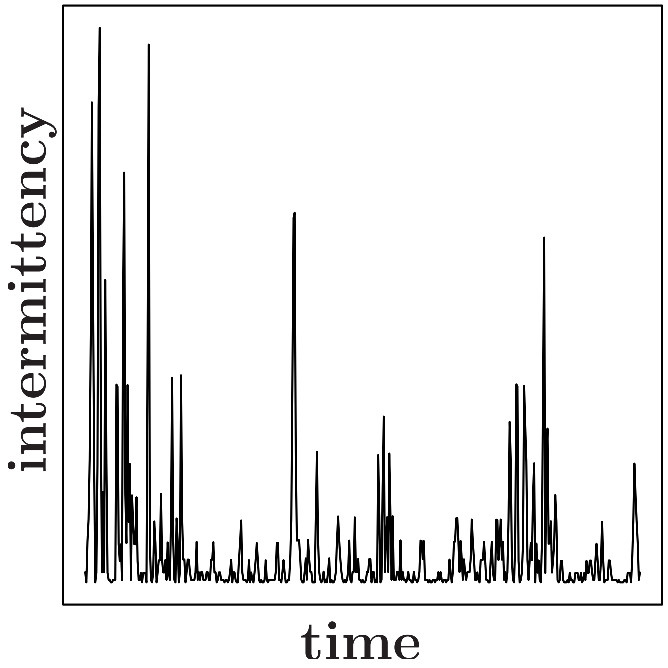
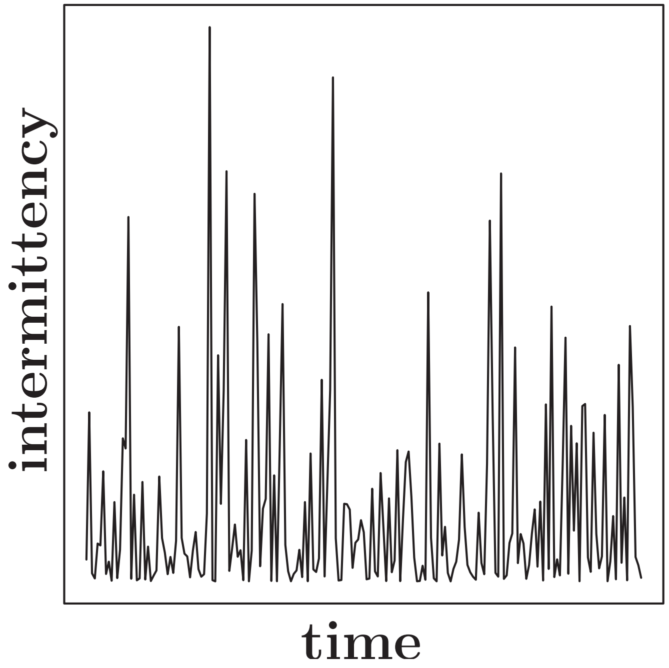
Time series: intermittency

$$J_t(\triangle t) = \frac{(v_{t+\triangle t} - v_t)^2}{\triangle t}$$

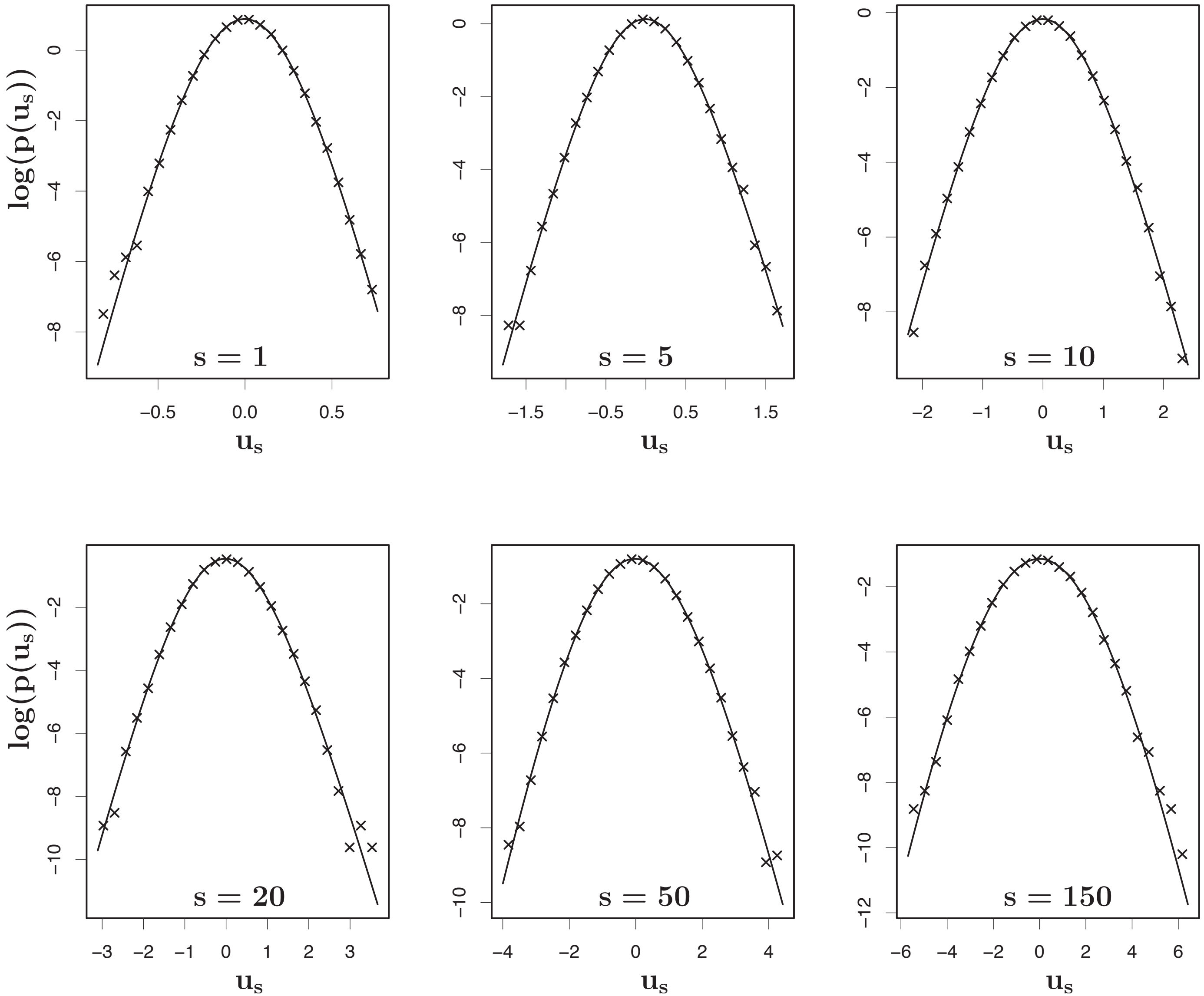
Data



Simulation



Densities of velocity increments



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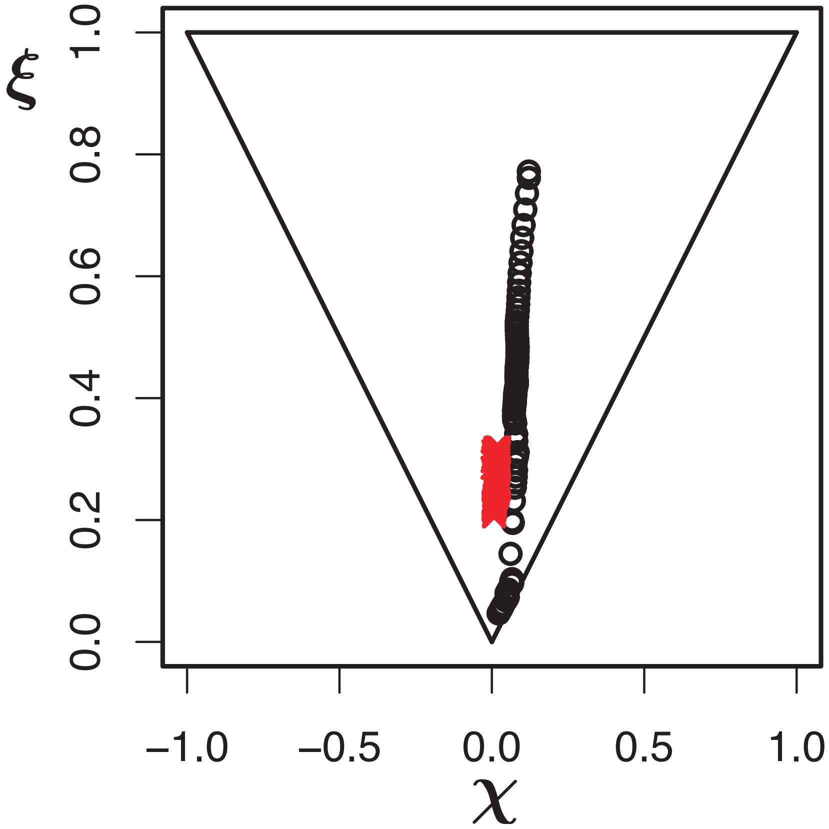
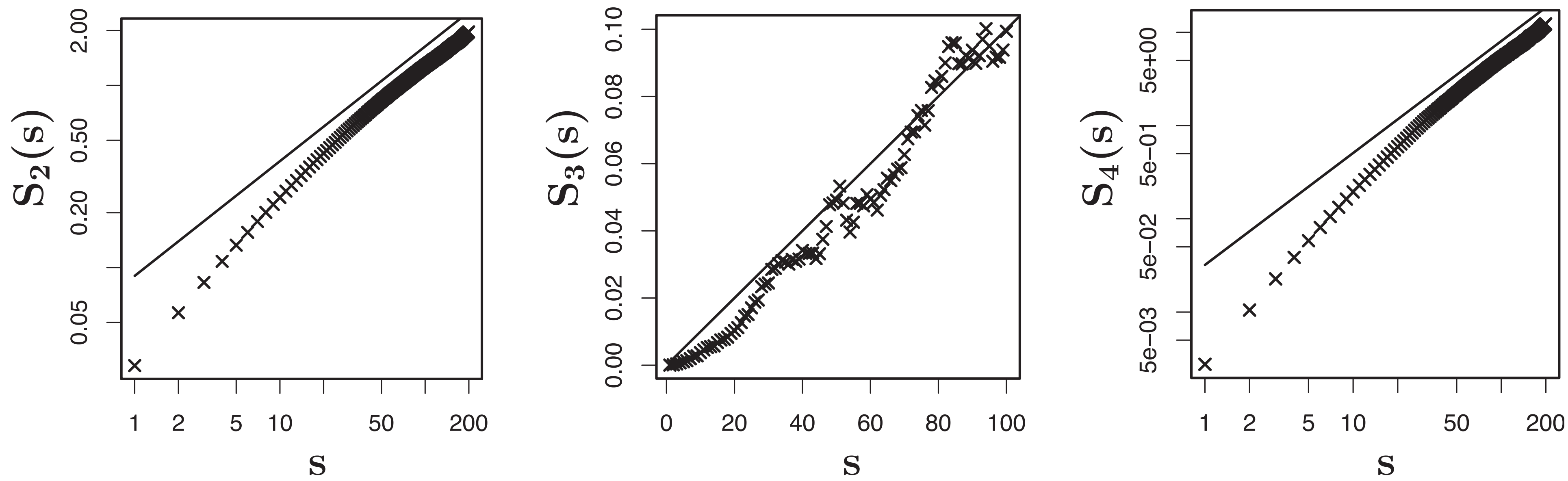
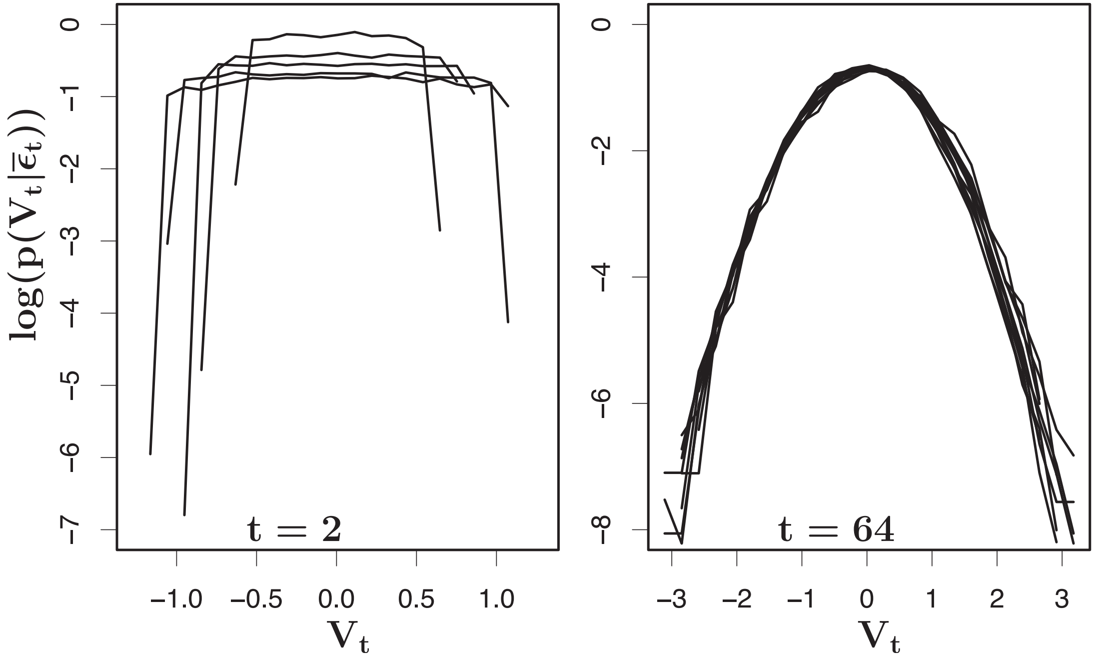


Scaling of structure functions



Kolmogorov variable



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Generalisation

We propose to model the spatio-temporal dynamics of the velocity $v_t(\sigma)$ at time t and position σ as an ambit process of the form

$$v_t(\sigma) = \int_{A_t(\sigma)} f(t-s,\sigma-\rho)\sqrt{J_s(\rho)}\,dZ + \beta \int_{B_t(\sigma)} g(t-s,\sigma-\rho)J_s(\rho)\,ds\,d\rho$$

where β is a constant, f and g are deterministic kernels,

$$\{A_t(\sigma) : (t,\sigma) \in \mathbb{R}^4\}$$

and

$$\{B_t(\sigma) : (t,\sigma) \in \mathbb{R}^4\}$$

are families of ambit sets and Z is a Lévy basis on $\mathbb{R}^4$.