Power Variation and Gaussian Processes with Stationary Increments

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Outline

1. Power variation for some class of processes
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   - Sequences of functionals of Gaussian processes
   - Gaussian processes with stationary increments
   - Integral processes
   - Convergence in probability
     - Ideas for the proof.
   - Central limit theorem I
     - Ideas for the proof
   - Central limit theorem II
     - Ideas for the proof

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   - Central limit theorem I
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2. More general functionals
   - Convergence in probability
     - Ideas for the proof
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Outline

For any $p > 0$, a natural number n and for any stochastic process $Z = \{Z_t, t \in [0, T]\}$ the (normalized) power variation of order p is defined as

$$V_p^n(Z)_t := \frac{1}{n\tau_n^p} \sum_{i=1}^{[nt]} \left| Z_{\frac{i}{n}} - Z_{\frac{i-1}{n}} \right|^p,$$

where τ_n is a normalization factor.

Consider a complete probability space $(\Omega, \mathcal{F}, P)$ and a Gaussian subspace $\mathcal{H}_1$ of $L^2(\Omega, \mathcal{F}, P)$ whose elements are zero-mean Gaussian random variables. Let be H a separable Hilbert space with scalar product denoted by $\langle \cdot, \cdot \rangle_H$ and norm $|| \cdot ||_H$, we will assume there is an isometry

$$
\begin{aligned}
W &: H \to \mathcal{H}_1 \\
h &\mapsto W(h)
\end{aligned}
$$

in the sense that

$$E(W(h_1)W(h_2)) = \langle h_1, h_2 \rangle_H.$$

For any $m \geq 2$, we denote by $\mathcal{H}_m$ the m-th Wiener chaos, that is, the closed subspace of $L^2(\Omega, \mathcal{F}, P)$ generated by the random variables $H_m(X)$, where $X \in \mathcal{H}_1$, $E(X^2) = 1$, and H_m is the m-th Hermite polynomial, i.e. $H_0(x) = 1$ and $H_m(x) = (-1)^m e^{\frac{x^2}{2}} \frac{d^m}{dx^m}(e^{-\frac{x^2}{2}})$.

Suppose that H is infinite-dimensional and let $\{e_i, i \geq 1\}$ be an orthonormal basis of H. Denote by Λ the set of all sequences $a = (a_1, a_2, ...)$, $a_i \in \mathbb{N}$, such that all the terms except a finite number of them, vanish. For $a \in \Lambda$ we set $a! = \Pi_{i=1}^{\infty} a_i!$ and $|a| = \sum_{i=1}^{\infty} a_i$. For any multiindex $a \in \Lambda$ we define

$$\Phi_a = \frac{1}{\sqrt{a!}} \Pi_{i=1}^{\infty} H_{a_i}(W(e_i)).$$

The family of random variables $\{\Phi_a, a \in \Lambda\}$ is an orthonormal system. In fact

$$E\left(\Pi_{i=1}^{\infty} H_{a_i}(W(e_i))\Pi_{i=1}^{\infty} H_{b_i}(W(e_i))\right) = \delta_{ab} a!.$$

And $\{\Phi_a, a \in \Lambda, |a| = m\}$ is a complete orthonormal system in $\mathcal{H}_m$.

Let $a \in \Lambda$, with $|a| = m$, the mapping

$$\begin{array}{rcl} I_m : H^{\odot m} & \to & \mathcal{H}_m \\ \widetilde{\otimes}_{i=1}^{\infty} e_i^{\otimes a_i} & \mapsto & \Pi_{i=1}^{\infty} H_{a_i}(W(e_i)), \end{array}$$

where $\widetilde{\otimes}$ denotes the symmetrization of the tensor product $\otimes$, between the symmetric tensor product $H^{\odot m}$, equipped with the norm $\sqrt{m!}\,\|\cdot\|_{H^{\otimes m}}$ and the m-th chaos, is a linear isometry. We also define I_0 as the identity in $\mathbb{R}$.

For any $h = h_1 \otimes \cdots \otimes h_m$ and $g = g_1 \otimes \cdots \otimes g_m \in H^{\otimes m}$, we define the p-th contraction of h and g, denoted by $h \otimes_p g$, as the element of $H^{\otimes 2(m-p)}$ given by

$$h \otimes_p g = < h_m, g_1 >_H \cdots < h_{m-p+1}, g_p >_H h_1 \otimes \cdots \otimes h_{m-p} \otimes g_{p+1} \otimes \cdots \otimes g_m.$$

This can be extended by linearity to any element of$H^{\otimes m}$. Note that if h and g belong to $H^{\odot m}$, $h \otimes_p g$ does not necessarily belong to $H^{\odot(2m-p)}$.

We have the following properties

- $I_m(e_i^{\otimes m}) = H_m(W(e_i))$.
- $I_p(h)I_q(g) = \sum_{r=0}^{p\wedge q} r! \begin{pmatrix} p \\ r \end{pmatrix} \begin{pmatrix} q \\ r \end{pmatrix} I_{p+q-2r}(h\widetilde{\otimes}_r g)$

Note that if we take $h = e_i^{\otimes p}, g = e_i^{\otimes q}$ we obtain

$$H_p(W(e_i))H_q(W(e_i)) = \sum_{r=0}^{p\wedge q} r! \binom{p}{r}\binom{q}{r} H_{p+q-2r}(W(e_i))$$

Theorem

Let $\mathcal{G}$ the σ-field generated by the random variables $\{W(h), h \in H\}$. Any square integrable random variable $F \in L^2(\Omega, \mathcal{G}, P)$ can be expanded as

$$F = \sum_{m=0}^{\infty} I_m(f_m).$$

Consider a sequence of d-dimensional random vectors $F_n = (F_n^1, F_n^2, ..., F_n^d)$, such that $F_n^k \in L^2(\Omega, \mathcal{G}, P)$ and

$$F_n^k = \sum_{m=0}^{\infty} I_m(f_{m,n}^k)$$

Theorem

Assume the following conditions hold:

(i) *For $k, l = 1, \ldots, d$ we have*

$$\lim_{n\to\infty} \sum_{m=1}^{\infty} m! ||f^k_{m,n}||^2_{H^{\otimes m}} = \Sigma_{kk}.$$

$$\sum_{m=1}^{\infty} \lim_{n\to\infty} \langle f^k_{m,n}, f^l_{m,n} \rangle = \Sigma_{kl}, \qquad k \neq l\,,$$

(ii) *For any $m \geq 1$, $k = 1, \ldots, d$ and $r = 1, \ldots, m-1$*

$$\lim_{n\to\infty} ||f^k_{m,n} \otimes_r f^k_{m,n}||^2_{H^{\otimes 2(m-r)}} = 0.$$

Then we have

$$F_n - f_{0,n} \xrightarrow{\mathcal{D}} N_d(0, \Sigma). \tag{1}$$

as n tends to infinity.

Consider the simple case where we have a family of stationary, centered, Gaussian random variables $\{X_i\}_{i\geq 1}$, with $E(X_1^2) = 1$, and we want to know the behavior of the sequence

$$Y_n = \frac{1}{\sqrt{n}}\left(\sum_{i=1}^{n} H(X_i) - E(H(X_1))\right)$$

when n goes to infinity. We assume that $E(H(X_1)^2) < \infty$. We can take $\mathcal{H}_1$ =span$\{X_i, i \geq 1\}$, and $H \equiv \mathcal{H}_1$!, then

$$H(x) = \sum_{m=0}^{\infty} c_m H_m(x)$$

and

$$\begin{aligned} Y_n &= \sum_{m=1}^{\infty} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} c_m H_m(X_i) \\ &= \sum_{m=1}^{\infty} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} c_m I_m(X_i^{\otimes m}) \\ &= \sum_{m=1}^{\infty} I_m\left(\frac{1}{\sqrt{n}} \sum_{i=1}^{n} c_m X_i^{\otimes m} \right) \end{aligned}$$

then

$$f_{m,n} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} c_m X_i^{\otimes m}$$

and

$$\sum_{m=1}^{\infty} m!||f_{m,n}||^2_{H^{\otimes m}} = \sum_{m=1}^{\infty} \frac{m!c_m^2}{n} \sum_{i,j=1}^{n} \rho(i-j)^m$$
$$= \sum_{m=1}^{\infty} m!c_m^2 \left(1 + \sum_{j=1}^{n-1} \rho(j)^m \left(1 - \frac{j}{n}\right)\right),$$

$$f_{m,n} \otimes_r f_{m,n} = \frac{c_m^2}{n} \sum_{i,j=1}^{n} \rho(i-j)^r X_i^{\otimes(m-r)} \otimes X_j^{\otimes(m-r)},$$

and

$$
\begin{aligned}
& \|f_{m,n} \otimes_r f_{m,n}\|^2_{H^{\otimes 2(m-r)}} \\
= \quad & \frac{c_m^4}{n^2} \sum_{i,j,k,l=1}^{n} \rho(i-j)^r \rho(k-l)^r \rho(i-k)^{m-r} \rho(j-l)^{m-r} \\
= \quad & \frac{c_m^4}{n} \sum_{i,j,k=0}^{n-1} \rho(i)^r \rho(j-k)^r \rho(j)^{m-r} \rho(i-k)^{m-r} (1 - \frac{i \vee j \vee k}{n})
\end{aligned}
$$

Let X be a centered Gaussian process X with stationary increments and such that

$$E(X_t - X_s)^2 = |t-s|^{2H} L(|t-s|), \qquad 0 < H < 1,$$

where L is a continuous function on $(0, \infty)$ slowly varying at zero, that is

$$\lim_{x \downarrow 0} \frac{L(tx)}{L(x)} = 1, \forall t > 0.$$

Since L is a continuous function slowly varying at zero, for any $\delta > 0$ and $x \in (0, 1]$ there exists a constant $K(\delta)$ such that

$$|L(x)| \leq K(\delta) x^{-\delta},$$

so

$$E(X_t - X_s)^2 \leq K(\delta)|t-s|^{2H-\delta}, 0 < s, t \leq 1$$

and then (a version of) X_t has trajectories $(H - \varepsilon)$-Hölder continuous.

Consequently the trajectories of X_t have p-finite variation with $p > 1/H$:

$$
\begin{aligned}
\mathrm{Var}_p(X;[a,b]) &= \sup_\pi \left(\sum_{i=1}^n \left| X_{t_i} - X_{t_{i-1}} \right|^p \right)^{1/p} \\
&\leq \|f\|_{H-\varepsilon} \sup_\pi \left(\sum_{i=1}^n \left| t_i - t_{i-1} \right|^{p(H-\epsilon)} \right)^{1/p} \\
&= \|f\|_{H-\varepsilon} \sup_\pi \left(\sum_{i=1}^n \left| t_i - t_{i-1} \right| \right)^{1/p} = \|f\|_{H-\varepsilon} (b-a)^{1/p},
\end{aligned}
$$

where $\|f\|_{H-\epsilon}$ is the Hölder norm and where we take $p = 1/(H-\varepsilon)$.

Young (1936) proved that the Riemann-Stieltjes integral $\int_a^b f dg$ exists if f and g do not have common discontinuities and they have finite p-variation and finite q-variation, respectively, in the interval $[a, b]$ and $\frac{1}{p} + \frac{1}{q} > 1$.
Then we can consider processes Z of the form

$$Z_t = \int_0^t u_s dX_s$$

where u is a process with finite q-variation $q < 1/(1 - H)$.
The purpose is to study asymptotic behavior of the power variation of Z

$$V_p^n(Z)_t = \frac{1}{n\tau_n^p} \sum_{i=1}^{[nt]} \left| \int_{\frac{i-1}{n}}^{\frac{i}{n}} u_s dX_s \right|^p$$

where $\tau_n = Var(X_{\frac{1}{n}})^{1/2}$.

Set $c_p = E(|N(0,1)|^p) = \frac{2^{p/2}\Gamma(\frac{p+1}{2})}{\Gamma(1/2)}$. Fix $T > 0$, denote by u.c.p. the uniform convergence in probability in the time interval $[0, T]$ and $\|\cdot\|_\infty$ for the supremum norm on $[0, T]$. Assume conditions

- C1 $t^{2H}L(t) \in C^2$ and $\left(t^{2H}L(t)\right)'' = t^{2H-2}L_1(t)$ where L_1 is slowly varying and continuous in $(0, \infty)$.
- C2 There exits b, $0 < b < 1$, such that

$$C = \limsup_{x \downarrow 0} \left(\sup_{\substack{y \\ x \le y \le x^b}} \left| \frac{L_1(y)}{L(x)} \right| \right) < \infty$$

then

Theorem

Suppose that $u = \{u_t, t \in [0, T]\}$ *is a stochastic process with finite q-variation, where* $q < \frac{1}{1-H}$. *Set*

$$Z_t = \int_0^t u_s dX_s.$$

Then,

$$V_p^n(Z)_t \xrightarrow{u.c.p} c_p \int_0^t |u_s|^p ds,$$

as n tends to infinity for any $t > 0$.

Assume first that $u_s \equiv 1$. Then $Z_t = X_t$ and

$$V_p^n(X)_t = \frac{1}{n\tau_n^p}\sum_{i=1}^{[nt]}\left|X_{\frac{i}{n}} - X_{\frac{i-1}{n}}\right|^p = \frac{1}{n}\sum_{i=1}^{[nt]}\left|\frac{X_{\frac{i}{n}} - X_{\frac{i-1}{n}}}{\tau_n}\right|^p,$$

therefore

$$E\left(V_p^n(Z)_t\right) = \frac{[nt]}{n}E(|N(0,1)|^p) = \frac{[nt]}{n}c_p$$

and

$$Var(V_p^n(X)_t) = \frac{[nt]}{n^2}(c_{2p}-c_p)+\frac{2}{n^2}\sum_{j=1}^{[nt]-1}(n-j)Cov(\left|\frac{X_{\frac{1}{n}}}{\tau_n}\right|^p,\left|\frac{X_{\frac{j+1}{n}} - X_{\frac{j}{n}}}{\tau_n}\right|^p)$$

Set

$$H(x) = |x|^p - c_p$$

we can write

$$H(x) = \sum_{m=2}^{\infty} a_m H_m(x)$$

where $H_m(x)$ are Hermite polynomials and $a_2 = \frac{pc_p}{2}$. Then

$$Cov(\left|\frac{X_{\frac{1}{n}}}{\tau_n}\right|^p, \left|\frac{X_{\frac{j+1}{n}} - X_{\frac{j}{n}}}{\tau_n}\right|^p) = \sum_{m=2}^{\infty} a_m^2 m! \rho_n(j)^m$$

where

$$\rho_n(j) = Cov(\frac{X_{\frac{1}{n}}}{\tau_n}, \frac{X_{\frac{j+1}{n}} - X_{\frac{j}{n}}}{\tau_n})$$

$$
\begin{aligned}
E(X_t X_s) &= \frac{1}{2}(E(X_t^2) + E(X_s^2) - E((X_t - X_s)^2)) \\
&= \frac{1}{2}(t^{2H}L(t) + s^{2H}L(s) - |t-s|^{2H}L(|t-s|))
\end{aligned}
$$

and consequently

$$
\rho_n(j) = \frac{1}{2L(\frac{1}{n})}\left((j+1)^{2H}L(\frac{j+1}{n}) + (j-1)^{2H}L(\frac{j-1}{n}) - 2j^{2H}L(\frac{j}{n})\right), j \geq 1
$$

By conditions **C1** and **C2**, and $m \geq 2$

$$\frac{1}{n}\sum_{j=1}^{[nt]-1}(1-\frac{j}{n})\rho_n(j)^m \sim \frac{1}{n}\sum_{j=1}^{[nt]-1}\rho(j)^m$$

with

$$\rho(j)=\frac{1}{2}\left((j+1)^{2H}+(j-1)^{2H}-2j^{2H}\right),j\geq 1.$$

Then

$$Var(V_p^n(X)_t)\underset{n\to\infty}{\to} 0$$

and therefore

$$V_p^n(Z)_t\xrightarrow{P} c_p t.$$

For the general case we can write, for any $m \geq n$ and if $p \leq 1$

$$
\begin{aligned}
& |V_p^m(Z)_t - c_p \int_0^t |u_s|^p ds| \\
\leq \; & \frac{1}{m\tau_m^p} \left| \sum_{j=1}^{[mt]} \left(\left| \int_{\frac{j-1}{m}}^{\frac{j}{m}} u_s dX_s \right|^p - \left| u_{\frac{j-1}{m}} (X_{\frac{j}{m}} - X_{\frac{j-1}{m}}) \right|^p \right) \right| \\
& + \frac{1}{m\tau_m^p} \left| \sum_{j=1}^{[mt]} \left| u_{\frac{j-1}{m}} (X_{\frac{j}{m}} - X_{\frac{j-1}{m}}) \right|^p - \sum_{i=1}^{[nt]} \left| u_{\frac{i-1}{n}} \right|^p \sum_{j \in I_n(i)} \left| X_{\frac{j}{m}} - X_{\frac{j-1}{m}} \right|^p \right| \\
& + \left| \frac{1}{m\tau_m^p} \sum_{i=1}^{[nt]} \left| u_{\frac{i-1}{n}} \right|^p \sum_{j \in I_n(i)} \left| X_{\frac{j}{m}} - X_{\frac{j-1}{m}} \right|^p - c_p n^{-1} \sum_{i=1}^{[nt]} \left| u_{\frac{i-1}{n}} \right|^p \right| \\
& + c_p \left| n^{-1} \sum_{i=1}^{[nt]} \left| u_{\frac{i-1}{n}} \right|^p - \int_0^t |u_s|^p ds \right| = \mathrm{A}_t^{(m)} + \mathrm{B}_t^{(n,m)} + \mathrm{C}_t^{(n,m)} + \mathrm{D}_t^{(n)},
\end{aligned}
$$

where $I_n(i) = \{j : \frac{j}{m} \in (\frac{i-1}{n}, \frac{i}{n}]\}, \; 1 \leq i \leq [nt]$.

For any fixed n, $\mathrm{C}_t^{(n,m)}$ converges in probability to zero, uniformly in t, as m tends to infinity. In fact,

$$\left\|\mathrm{C}^{(n,m)}\right\|_\infty \leq \sum_{i=1}^{[nT]} \left|u_{\frac{i-1}{n}}\right|^p \left|\frac{1}{m\tau_m^p} \sum_{j\in I_n(i)} \left|X_{\frac{j}{m}} - X_{\frac{j-1}{m}}\right|^p - c_p n^{-1}\right|$$

and we can use the result for $u \equiv 1$. In a similar way we can prove that as m tends to infinity

$$\limsup_m \left\|\mathrm{B}^{(n,m)}\right\|_\infty \leq \frac{c_p}{n} \sum_{i=1}^{[nT]} \sup_{s\in\mathcal{I}_n(i)\cup\mathcal{I}_n(i-1)} \left||u_{\frac{i-1}{n}}|^p - |u_s|^p\right| + \||u|^p\|_\infty ,$$

where $\mathcal{I}_n(i) = \left[\frac{i-1}{n}, \frac{i}{n}\right]$, and this tends to zero as n goes to infinity because $|u|^p$ is *regulated*.

$$
\begin{aligned}
\left\| \mathrm{A}^{(n,m)} \right\|_\infty &\leq \frac{1}{m\tau_m^p} \sum_{j=1}^{[mt]} \left| \int_{\frac{j-1}{m}}^{\frac{j}{m}} u_s dX_s - u_{\frac{j-1}{m}} (X_{\frac{j}{m}} - X_{\frac{j-1}{m}}) \right|^p \\
&\leq c_{\frac{1}{H-\varepsilon},q} \frac{1}{m\tau_m^p} \sum_{j=1}^{[mT]} \Big(\mathrm{Var}_q(u; \mathcal{I}_m(j)) \mathrm{Var}_{\frac{1}{H-\varepsilon}}(X; \mathcal{I}_m(j)) \Big)^p \\
&\leq c_{\frac{1}{H-\varepsilon},q} T ||u||_q^p ||X||_{\frac{1}{H-\varepsilon}}^p m^{-pq+p\varepsilon}
\end{aligned}
$$

For $H \in (0, \frac{3}{4})$ the fluctuations of the power variation, properly normalized, have Gaussian asymptotic distributions. Write

$$\sigma_m^2 = 1 + 2\sum_{j=1}^{\infty} \rho(j)^m$$

with

$$\rho(j) = \frac{1}{2}\left((j+1)^{2H} + (j-1)^{2H} - 2j^{2H}\right), j \geq 1.$$

and

$$\sigma^2 = \sum_{m=2}^{\infty} a_m^2 m! \sigma_m^2$$

Theorem

Fix $p > 0$. Assume $0 < H < 3/4$. Then

$$(X_t, \sqrt{n}(\ V_p^n(X)_t - c_p t) \xrightarrow{\mathcal{L}} (X_t, \sigma W_t), \tag{2}$$

as n tends to infinity, where $W = \{W_t, t \in [0, T]\}$ is a Brownian motion independent of the process X, and the convergence is in the space $\mathcal{D}([0, T])^2$ equipped with the Skorohod topology.

The proof has two steps. Set

$$Z_t^{(n)} = \sqrt{n}(\ V_p^n(X)_t - c_p t)$$

Step 1. First we have to show the convergence of the finite dimensional distributions. Let $J_k = (a_k, b_k]$, $k = 1, \ldots, N$ be pairwise disjoint intervals contained in $[0, T]$. Define the random vectors $X = (X_{b_1} - X_{a_1}, \ldots, X_{b_N} - X_{a_N})$ and $Y^{(n)} = (Y_1^{(n)}, \ldots, Y_N^{(n)})$, where

$$Y_k^{(n)} = \frac{1}{\sqrt{n}} \sum_{[na_k]<j\le[nb_k]} \left| \frac{X_{j/n} - X_{(j-1)/n}}{\tau_n} \right|^p - \sqrt{n} c_p |J_k|,$$

$k = 1, \ldots, N$ and $|J_k| = b_k - a_k$. We have to show that

$$(X, Y^{(n)}) \xrightarrow{\mathcal{L}} (X, V), \tag{3}$$

where X and V are independent and V is a Gaussian random vector with zero mean, and independent components of variances $\sigma^2|J_k|$.

Set $X_{j,n} = \frac{X_{j/n} - X_{(j-1)/n}}{\tau_n}$ and $H(x) = |x|^p - c_p$. Then, $\{X_{j,n}, j \geq 1\}$ is a stationary Gaussian triangular system with zero mean, unit variance and $E(X_{1,n} X_{j+1,n}) = \rho_n(j)$.Thus, the convergence (3) is equivalent to the convergence in distribution of $(X^{(n)}, Y^{(n)})$ to (X, V), where

$$X_k^{(n)} = \tau_n \sum_{[na_k] < j \leq [nb_k]} X_{j,n}, \; 1 \leq k \leq N \tag{4}$$

and

$$Y_k^{(n)} = \frac{1}{\sqrt{n}} \sum_{[na_k] < j \leq [nb_k]} H(X_{j,n}), \; 1 \leq k \leq N. \tag{5}$$

Then we can take $\mathcal{H}_1$ to be the closed subspace of $L^2(\Omega, \mathcal{F}, P)$ generated by the random variables $X_{n,j}$. and to apply the results on sequences of functionals of Gaussian processes mentioned before.

Step 2. We need to show that the sequence of processes $Z^{(n)}$ is tight in $\mathcal{D}([0,T])$. Let us compute for $s<t$

$$E\left(\left|Z_t^{(n)}-Z_s^{(n)}\right|^4\right)=n^{-2}E\left(\left|\sum_{j=[ns]+1}^{[nt]}H(X_{j,n})\right|^4\right).$$

If H is polynomial we have that, for all $N\geq 1$

$$\frac{1}{N^2}E\left(\left|\sum_{j=1}^{N}H(X_{j,n})\right|^4\right)\leq K,$$

this is guaranteed by the behavior of the contractions mentioned before, then for all $t_1\leq t\leq t_2$

$$E\left(\left|Z_{t_2}^{(n)}-Z_t^{(n)}\right|^2\left|Z_t^{(n)}-Z_{t_1}^{(n)}\right|^2\right)\leq C\left|t_2-t_1\right|^2,$$

and by Billingsley (1968, Theorem 15.6) we get the desired tightness property, finally, for general H, we can use an approximation argument.

Theorem

Fix $p > 0$. *Let* $H \in (0, 3/4)$. *Suppose that* $u = \{u_t, t \in [0, T]\}$ *is a stochastic process, measurable with respect to* X, *with Hölder continuous trajectories of order* $a > \frac{1}{2(p \wedge 1)}$. *Set* $Z_t = \int_0^t u_s dX_s$. *Then*

$$\left(X_t, \sqrt{n}(V_p^n(Z)_t - c_p \int_0^t |u_s|^p ds)\right) \xrightarrow{\mathcal{L}} \left(X_t, \sigma \int_0^t |u_s|^p dW_s\right),$$

as n tends to infinity, where $W = \{W_t, t \in [0, T]\}$ *is a Brownian motion independent of* X *and the convergence is in* $\mathcal{D}([0, T])^2$.

For any $m \geq n$, we can write,

$$\sqrt{m}(V_p^m(Z)_t - c_p \int_0^t |u_s|^p ds) = A_t^{(m)} + B_t^{(n,m)} + C_t^{(n,m)} + D_t^{(m)},$$

where

$$A_t^{(m)} = \frac{1}{\sqrt{m}\tau_n^p} \sum_{j=1}^{[mt]} \left(\left| \int_{\frac{j-1}{m}}^{\frac{j}{m}} u_s dX_s \right|^p - \left| u_{\frac{j-1}{m}} (X_{\frac{j}{m}} - X_{\frac{j-1}{m}}) \right|^p \right),$$

$$
\begin{aligned}
B_t^{(n,m)} &= \frac{1}{\sqrt{m}\tau_n^p}\sum_{j=1}^{[mt]}\left|u_{\frac{j-1}{m}}(X_{\frac{j}{m}}-X_{\frac{j-1}{m}})\right|^p - \frac{1}{\sqrt{m}}c_p\sum_{j=1}^{[mt]}|u_{\frac{j-1}{m}}|^p \\
&\quad -\sum_{i=1}^{[nt]}|u_{\frac{i-1}{n}}|^p\sum_{j\in I_n(i)}\frac{1}{\sqrt{m}\tau_n^p}\left|X_{\frac{j}{m}}-X_{\frac{j-1}{m}}\right|^p + \frac{\sqrt{m}}{n}c_p\sum_{i=1}^{[nt]}|u_{\frac{i-1}{n}}|^p ,
\end{aligned}
$$

$$
C_t^{(n,m)} = \sum_{i=1}^{[nt]}|u_{\frac{i-1}{n}}|^p\left(\sqrt{m}(\sum_{j\in I_n(i)}\frac{1}{m\tau_n^p}\left|X_{\frac{j}{m}}-X_{\frac{j-1}{m}}\right|^p-\frac{c_p}{n})\right)
$$

and

$$
D_t^{(m)} = \frac{1}{\sqrt{m}}c_p\sum_{j=1}^{[mt]}|u_{\frac{j-1}{m}}|^p - \sqrt{m}c_p\int_0^t|u_s|^p ds.
$$

Then as $m \to \infty$

$$C_t^{(n,m)} \underset{\text{stably}}{\overset{\mathcal{L}}{\to}} \sigma \sum_{i=1}^{[nt]} |u_{\frac{i-1}{n}}|^p \left(W_{\frac{i}{n}} - W_{\frac{i-1}{n}} \right)$$

and as $n \to \infty$

$$\sum_{i=1}^{[nt]} |u_{\frac{i-1}{n}}|^p \left(W_{\frac{i}{n}} - W_{\frac{i-1}{n}} \right) \overset{u.c.p}{\to} \int_0^t |u_s|^p dW_s$$

Let $Z = \{Z_t, t \geq 0\}$ be a stochastic process, define

$$Z_t^{(n)} = Z_{\frac{i-1}{n}} + n(t - \frac{i-1}{n})(Z_{\frac{i}{n}} - Z_{\frac{i-1}{n}}), \frac{i-1}{n} \leq t < \frac{i}{n}, \tag{6}$$

that is the broken line approximation of Z_t. The derivative of $Z_t^{(n)}$, that we denote by $\dot{Z}_t^{(n)}$, is defined except for a finite number of points. We are going to study the asymptotic behavior of functionals of the form

$$F_{g,h}^{(n)}(Z)_t = \int_0^{\frac{[nt]}{n}} h(Z_s^{(n)}) g\left(\dot{Z}_s^{(n)} n^{H-1}\right) ds, \tag{7}$$

where h and g are continuous functions.

In the particular case $g(x) = |x|^p$, where $p > 0$, and $h \equiv 1$, $F^{(n)}_{g,h}(Z)$ is the normalized power variation of order p that we will write $V^{(n)}_g(Z)_t$. In fact

$$\int_0^{\frac{[nt]}{n}} \left| \dot{Z}^{(n)}_s n^{H-1} \right|^p ds = \frac{1}{n} \sum_{i=1}^{[nt]} \left| (Z_{\frac{i}{n}} - Z_{\frac{i-1}{n}}) n^H \right|^p$$

These kind of functionals have been considered in Leon and Ludeña (2004). There the authors study their asymptotic behavior by assuming that Z is the solution of an stochastic differential equation driven by a fBm with $H > 1/2$.

We are going to impose the following condition on the function g:
H: There exist constants $\alpha \in (0,1]$, $a, b \geq 0$ and $0 \leq p < 2$ such that for all $0 \leq x < y$ we have

$$|g(y) - g(x)| \leq C(\xi)|y - x|^{\alpha},$$

where $\xi \in [x, y]$ and the function C satisfies $0 \leq C(u) \leq ae^{b|u|^p}$.
We will denote by W a standard normal random variable independent of the process B^H, and E^W will denote the mathematical expectation with respect to W. Let $c_g(z) = E^W(g(zW))$ for any $z > 0$.

Theorem

Suppose that $u = \{u_t, t \in [0, T]\}$ *is an stochastic process with finite q-variation, where* $q < \frac{1}{1-H}$. *Set*

$$Z_t = \int_0^t u_s dB_s^H.$$

Then,

$$F_{g,h}^{(n)}(Z)_t \xrightarrow{u.c.p} \int_0^t h(Z_s) c_g(u_s) ds,$$

as n tends to infinity.

For any $m \geq n$,

$$\begin{aligned}
&\left| F_{g,h}^{(m)}(Z)_t - \int_0^t h(Z_s) E^W \left(g(u_s W) \right) ds \right| \\
&\leq \left| \sum_{j=1}^{[mt]} \int_{\frac{j-1}{m}}^{\frac{j}{m}} h(Z_r^{(m)}) dr \left(g\left(m^H \int_{\frac{j-1}{m}}^{\frac{j}{m}} u_s dB_s^H \right) - g\left(m^H u_{\frac{j-1}{m}} \Delta B_{\frac{j}{m}}^H \right) \right) \right| \\
&+ \left| \sum_{j=1}^{[mt]} \left(\int_{\frac{j-1}{m}}^{\frac{j}{m}} h(Z_r^{(m)}) dr \right) g\left(m^H u_{\frac{j-1}{m}} \Delta B_{\frac{j}{m}}^H \right) - \frac{1}{m} \sum_{i=1}^{[nt]} \sum_{j \in I(i)} h(Z_{\frac{i-1}{n}}) g\left(m^H u_{\frac{i-1}{n}} \Delta B_{\frac{j}{m}}^H \right) \right| \\
&+ \left| \frac{1}{m} \sum_{i=1}^{[nt]} \sum_{j \in I(i)} h(Z_{\frac{i-1}{n}}) g\left(m^H u_{\frac{i-1}{n}} \Delta B_{\frac{j}{m}}^H \right) - \frac{1}{n} \sum_{i=1}^{[nt]} h(Z_{\frac{i-1}{n}}) E^W \left(g(u_{\frac{i-1}{n}} W) \right) \right| \\
&+ \left| \frac{1}{n} \sum_{i=1}^{[nt]} h(Z_{\frac{i-1}{n}}) E^W (g\left(u_{\frac{i-1}{n}} W \right)) - \int_0^t h(Z_s) E^W (g\left(u_s W \right)) ds \right| \\
&= A_t^{(m)} + B_t^{(n,m)} + C_t^{(n,m)} + D_t^{(n)},
\end{aligned}$$

where for each $i = 1, \ldots, n$, $I(i) = \{ j : \frac{j}{m} \in (\frac{i-1}{n}, \frac{i}{n}] \}$.

For any fixed n, $\left\|C^{(n,m)}\right\|_\infty$ converges in probability to zero as m tends to infinity, by the ergodic theorem on Banach spaces. In fact, fix n and for any constant K define

$$Y_{m,K} = \frac{1}{m} \sum_{j \in I(i)} g\left(m^H K(B^H_{\frac{j}{m}} - B^H_{\frac{j-1}{m}})\right),$$

and

$$Z_{m,K} = \frac{1}{m} \sum_{\frac{(i-1)m}{n} < j \le \frac{im}{n}} g\left(K(B^H_j - B^H_{j-1})\right).$$

By the self-similarity of the fractional Brownian motion, and for any constant $M > 0$ the family of random variables

$$\left\{Y_{m,K}, K \in [-M, M], m \ge 1\right\}$$

has the same distribution as

$$\left\{Z_{m,K}, K \in [-M, M], m \ge 1\right\}.$$

Let $C([-M, M], \mathbb{R})$ be the Banach space of continuous functions from $[-M, M]$ to $\mathbb{R}$ with the supremum norm. Then $\{g\left(|\cdot(B_j^H - B_{j-1}^H)|\right), j \geq 1\}$, is a stationary sequence with values in $C([-M, M], \mathbb{R})$. Then, by the ergodic theorem in Banach spaces and the uniqueness of the L^1 limit we have that

$$E\left(\sup_{K\in[-M,M]}\left|Z_{m,K} - \frac{1}{n}E^W(g(KW))\right|\right) \to 0$$

as m tends to infinity.

As a consequence we can write

$$P\left(\left|Y_{m,u_{\frac{i-1}{n}}}-\frac{1}{n}E^{W}\left(g(u_{\frac{i-1}{n}}W)\right)\right|>\delta\right)$$
$$\leq P\left(\left|Y_{m,u_{\frac{i-1}{n}}}-\frac{1}{n}E^{W}\left(g(u_{\frac{i-1}{n}}W)\right)\right|>\delta,\|u\|_{\infty}\leq M\right)+P\left(\|u\|_{\infty}>M\right).$$

The second summand in the above expression converges to zero as M tends to infinity. The first one is bounded by

$$P\left(\sup_{K\in[-M,M]}\left|Y_{m,K}-\frac{1}{n}E^{W}\left(g(KW)\right)\right|>\delta\right)$$
$$\leq\frac{1}{\delta}E\left(\sup_{K\in[-M,M]}\left|Z_{m,K}-\frac{1}{n}E^{W}(g\left(KW\right))\right|\right),$$

which converges to zero as m tends to infinity.

For the term $A_t^{(m)}$ we have to use the properties of the modulus of continuity of the fractional Brownian, that imply the existence of a finite random variable G_1 such that if $|t-s| \leq \frac{1}{2}$,

$$|B_t^H - B_s^H| \leq G_1 |t-s|^H \sqrt{\log |t-s|^{-1}}.$$

Fernique's theorem and the ergodic theorem.

Let $\sigma^2(z) = \lim_{n\to\infty} Var\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n} g(z(B_i^H - B_{i-1}^H))\right)$. Assume that $H < 3/4$ and that g is an even function.
Then

Theorem

$$(B_t^H, \sqrt{n}(V_g^{(n)}(zB^H)_t - c_g(z)t)) \xrightarrow{\mathcal{L}} (B_t^H, \sigma(z)W_t), \tag{8}$$

as n tends to infinity, where $W = \{W_t, t \in [0, T]\}$ *is a Brownian motion independent of the process* B^H*, and the convergence is in the space* $\mathcal{D}([0, T])^2$ *equipped with the Skorohod topology.*

Theorem

Suppose that $u = \{u_t, t \in [0, T]\}$ *is a stochastic process measurable with respect to* B^H, *with Hölder continuous trajectories of order* $a > \frac{1}{2\alpha}$. *Set* $Z_t = \int_0^t u_s dB_s^H$, *with* $1/2 < H < 3/4$. *Assume the function* h *is Lipschitz and verifies also condition* ***H.*** *Then*

$$\left(B_t^H, \sqrt{n}(F_{g,h}^{(n)}(Z)_t - \int_0^t h(Z_s)c_g(u_s)ds) \right) \xrightarrow{\mathcal{L}} \left(B_t^H, \int_0^t h(Z_s)\sigma(u_s)dW_s \right),$$

as n *tends to infinity, where* $W = \{W_t, t \in [0, T]\}$ *and the convergence is in* $\mathcal{D}([0, T])^2$.

- Corcuera, J. M., Nualart, D. and Woerner, J. H. C. (2006) Power variation of some integral fractional processes. *Bernoulli,* **12**(4), 713-735.
- Corcuera, J. M., Nualart, D. and Woerner, J. H. C (2007). A functional Central Limit Theorem for the Realized Power Variation of Integrated Stable Processes. *Stochastic Analysis and Applications*, 25, 169-186.
- Corcuera, J. M., Nualart, D. and Woerner, J. H. C (2007). Convergence of certain functionals of integral fractional processes. Preprint.

Barndorff-Nielsen, O. E., Corcuera J. M. and Podolskij, M. (2007). Power variation for Gaussian processes with independent increments. Preprint.

Guyon, X. and Leon, J. (1989). Convergence en loi des H-variations d'un process gaussien stationnaire sur $\mathbb{R}$. *Annales de l' Institut Henri Poincaré*, 25(3), 265-282.

Leon, J. and Ludeña, C. (2004). Stable Convergence of Certain Functionals Driven by fBm. *Stochastic Analysis and Applications*, 22, 289-314.