- new insights into turbulence with excursion to finance -

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ForWind
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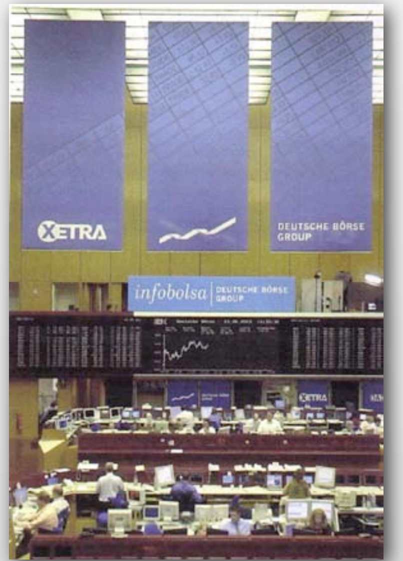
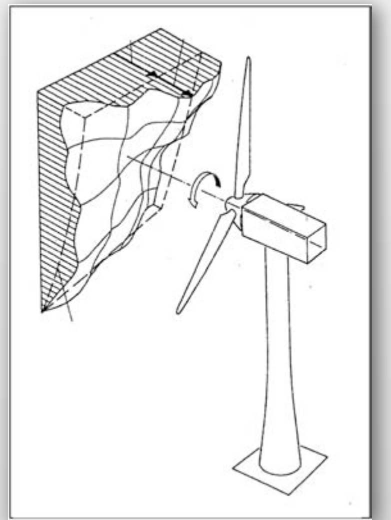
research topics

in finance



data analysis

wind energy

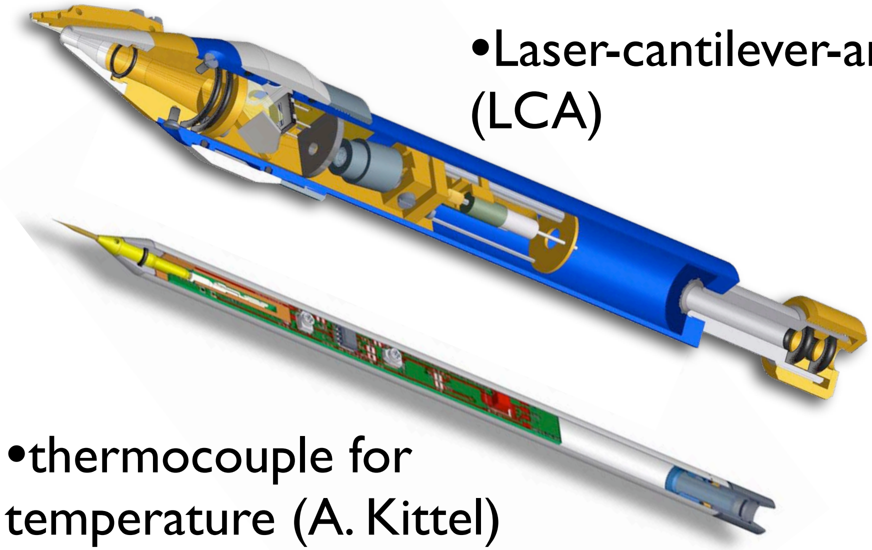


real turbulence

lokal isotrope turbulence - experiment

- at least we can measure the turbulence

•hot wire anemometer



•Laser-cantilever-anemometer (LCA)

•thermocouple for temperature (A. Kittel)

turbulence

open question: to understand the correlations of the disorder of the turbulent field

$$\left\langle u_i^{\alpha}(x)\cdot u_j^{\beta}(x+r)\right\rangle$$

for r=>0 Reynolds stress

alternatively increments for spatial correlations

$$\vec{u}_r(x) = \vec{u}(x+r) - \vec{u}(x)$$

with u_r longitudinal and
v_r transversal increments

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statistics of turbulence

challenge to know - general n-scale statistics

$$p(\vec{u}_1, r_1;\ \vec{u}_2, r_2; \ldots;\ \vec{u}_n, r_n)$$

$$\left\langle \vec{u}_1^{\ \alpha_1} \cdot \vec{u}_2^{\ \alpha_2} \ \ldots\ \vec{u}_n^{\ \alpha_n} \right\rangle$$

Known is

Kolmogorov $\left\langle u_r^{\ 3} \right\rangle = -\frac{4}{5}\varepsilon_r r + 6\nu \frac{\partial}{\partial r}\left\langle u_r^2 \right\rangle$

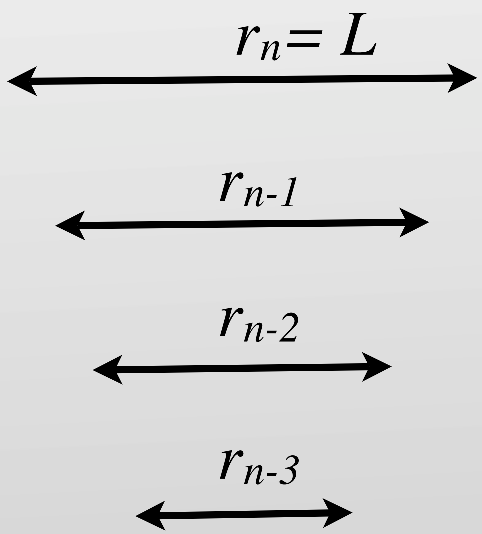
$$\left\langle u_r(x)^n \right\rangle \propto C_n r^{\xi_n}$$

Karman $-r\frac{\partial}{\partial r}\left\langle u_r^{\ 2} \right\rangle = 2\left\langle u_r^{\ 2} \right\rangle - 2\left\langle v_r^{\ 2} \right\rangle$

statistics of turbulence

n-scale statistics

$$p(\vec{u}_1, r_1;\ \vec{u}_2, r_2; \ldots;\ \vec{u}_n, r_n)$$



what are possible simplifications?

all increments at the same location

$$u_{r_i} =: u_i = u(x + r_i) - u(x)$$

statistics of turbulence -2-

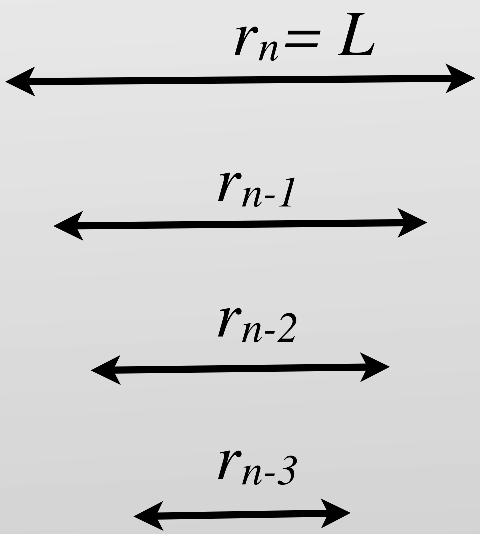
n-scale statistics

$$p(\vec{u}_1, r_1;\ \vec{u}_2, r_2; \ldots;\ \vec{u}_n, r_n)$$

what are possible simplifications?

formula of Bayes

$$p(\vec{u}_1, r_1; \ldots;\ \vec{u}_n, r_n) =$$
$$p(\vec{u}_1, r_1 \,|\, \vec{u}_2, r_2; \ldots;\ \vec{u}_n, r_n)\, p(\vec{u}_2, r_2; \ldots;\ \vec{u}_n, r_n)$$



simplification if:

$$p(\vec{u}_1, r_1 \,|\, \vec{u}_2, r_2; \ldots;\ \vec{u}_n, r_n) = p(\vec{u}_1, r_1 \,|\, \vec{u}_2, r_2)$$

or

$$p(\vec{u}_1, r_1 \,|\, \vec{u}_2, r_2; \ldots;\ \vec{u}_n, r_n) = p(\vec{u}_1, r_1)$$

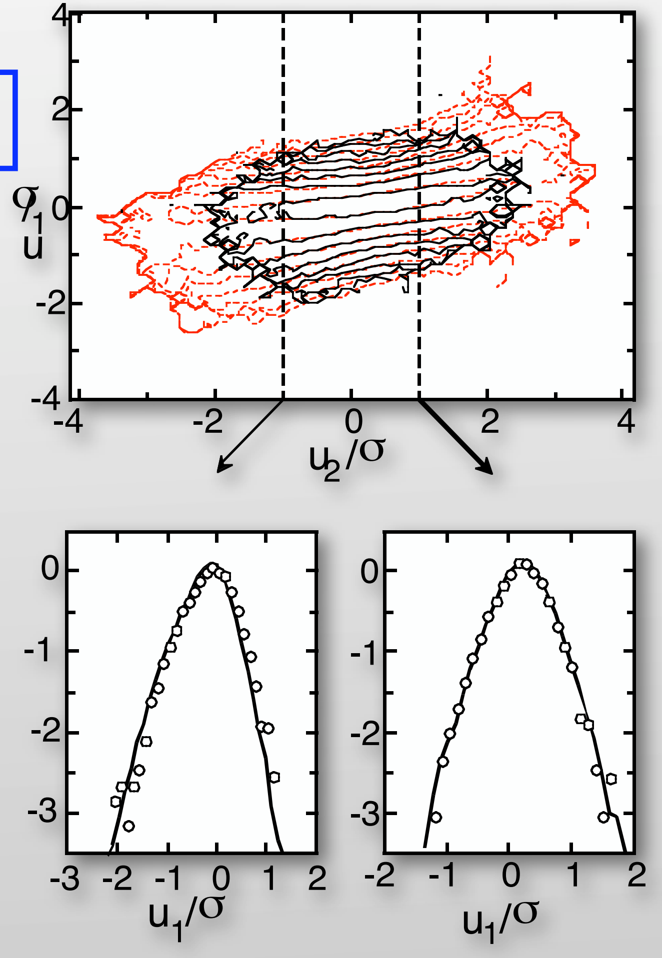
statistics of turbulence -3-

simplification

(1) $$p(\vec{u}_1, r_1 | \vec{u}_2, r_2; \ldots; \vec{u}_n, r_n) = p(\vec{u}_1, r_1 | \vec{u}_2, r_2)$$

(2) $$p(\vec{u}_1, r_1 | \vec{u}_2, r_2; \ldots; \vec{u}_n, r_n) = p(\vec{u}_1, r_1)$$

experimental test



experimental result:

$$p(u_1|u_2,u_3) = p(u_1|u_2)$$

(1) holds

(2) not

statistics of turbulence -4-

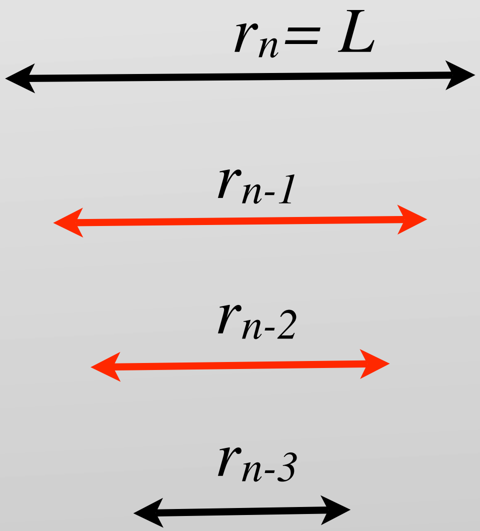
general n-scale statistics can be expressed by

$$p(\vec{u}_1, r_1; \ldots; \vec{u}_n, r_n) = p(\vec{u}_1, r_1 | \vec{u}_2, r_2)\, p(\vec{u}_2, r_2 | \vec{u}_3, r_3) \ldots p(\vec{u}_{n-1} | \vec{u}_n)\, p(\vec{u}_n, r_n)$$

and not

$$p(\vec{u}_1, r_1; \ldots; \vec{u}_n, r_n) \neq p(\vec{u}_1, r_1)\, p(\vec{u}_2, r_2) \ldots p(\vec{u}_n, r_n)$$

with cascades picture

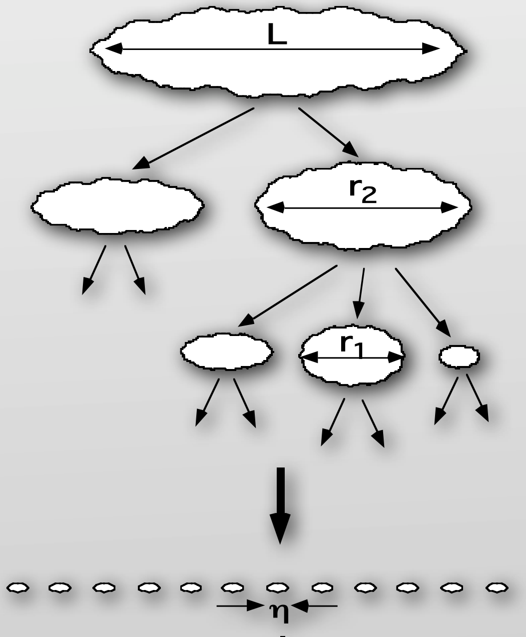


Cascade a Markov process

stochastic cascade process

idea of a turbulent cascade:

large vortices are generating small ones



$$\partial_r u_r$$

$$\partial_r p_r(u_r)$$

=> stochastic cascade process evolving in r

stochastic cascade process - 2 -

summary: characterization of the disorder by joint n-scale statistics by a stochastic process,

1. proof of Markov properties

$$p(\vec{u}_1, r_1 | \vec{u}_2, r_2; \ldots; \vec{u}_n, r_n) = p(\vec{u}_1, r_1 | \vec{u}_2, r_2)$$

2. estimation of the Kramers Moyal coefficients results in simplification:

$$D^{(n)}(u,r) = \lim_{\Delta r \to 0} \frac{1}{n! \cdot \Delta r} \int (\tilde{u} - u)^n \, p(\tilde{u}, r - \Delta r \mid u, r) \, d\tilde{u}$$

3. obtain information for the n-scale statistics by process equation (Fokker-Planck or Kolomogorov equation)

$$-\frac{\partial}{\partial r} p(u\ ,r | u_0, r_0) = \left[-\frac{\partial}{\partial u} D^{(1)}(u,r) + \frac{\partial^2}{\partial u^2} D^{(2)}(u,r) \right] \cdot p(u\ ,r | u_0, r_0)$$

stochastic cascade process -3-

1. property of a Markov process:

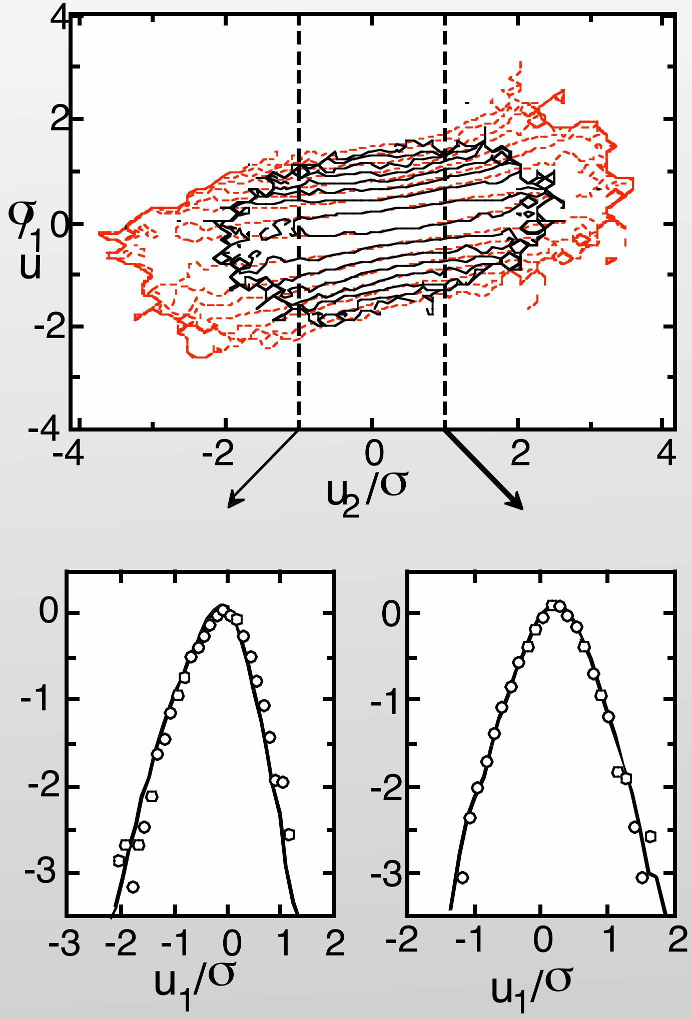
- evidence by conditional

  probability densities

$$p(u_1|u_2, \ldots, u_N) = p(u_1|u_2)$$

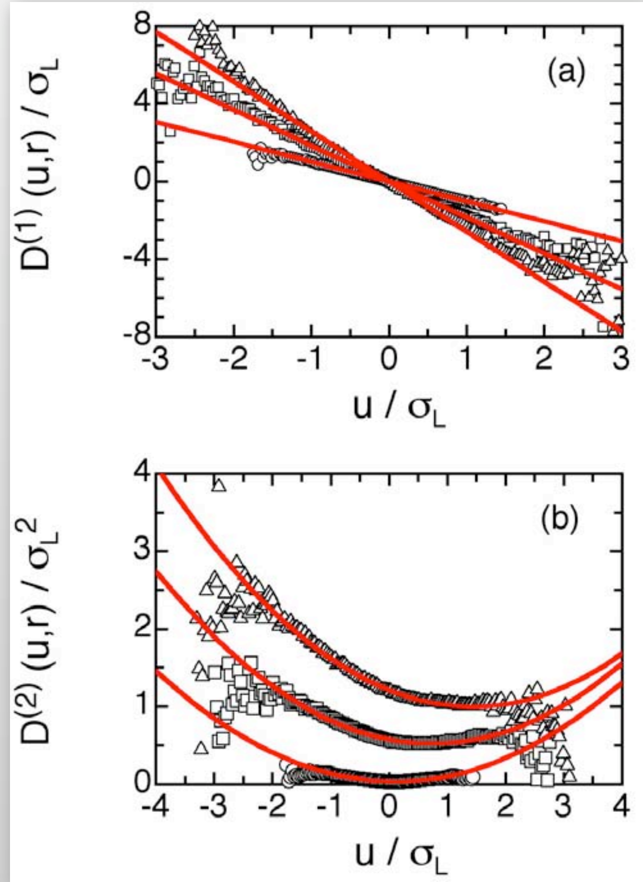
- experimental result:

$$p(u_1|u_2,u_3) = p(u_1|u_2)$$



stochastic cascade process -4-

2. measured: $D^{(1)}(u,r)$ and $D^{(2)}(u,r)$



$D^{(1)}(u,r) \cong \gamma(r)\, u(r)$

$D^{(2)}(u,r) \cong \alpha(r) + \delta(r)\, u(r) + \beta(r)\, u^2(r)$

with the definition of (after Kol. 1931)

$$D^{(k)}(u,r) = \lim_{\Delta r \to 0} \frac{r}{k!\Delta r} M^{(k)}(u,r,\Delta r),$$

$$M^{(k)}(u,r,\Delta r) = \int_{-\infty}^{+\infty} (\tilde{u}-u)^k p(\tilde{u}, r-\Delta r | u, r)\, d\tilde{u}$$

Phys. Rev. Lett. **89**, (2002)

stochastic cascade process -5-

measured Fokker-Planck equation

$$-\frac{\partial}{\partial r}\left\langle u_r^{\,n}\right\rangle = n\cdot\left\langle u_r^{\,n-1} D^{(1)}(u_r,r)\right\rangle + n\cdot(n-1)\left\langle u_r^{\,n-2} D^{(2)}(u_r,r)\right\rangle$$

- closed equation for structure functions if

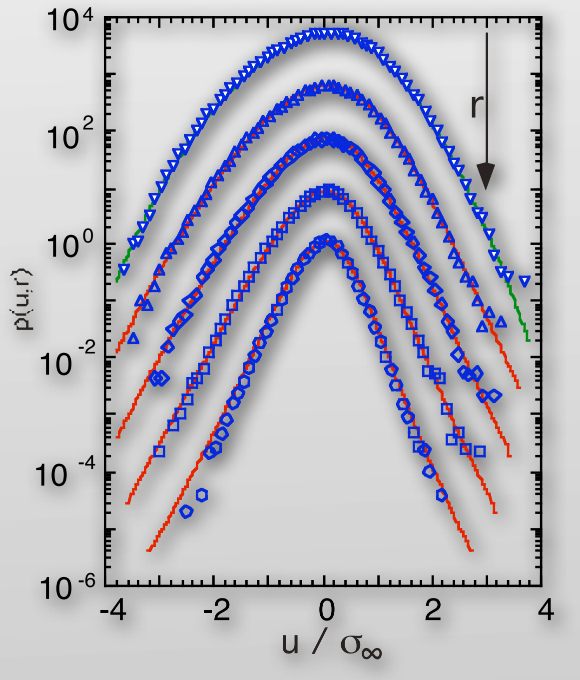
$D^{(1)}(u,r) \cong \gamma(r)\, u(r)$

$D^{(2)}(u,r) \cong \alpha(r) + \delta(r)\, u(r) + \beta(r)\, u^2(r)$

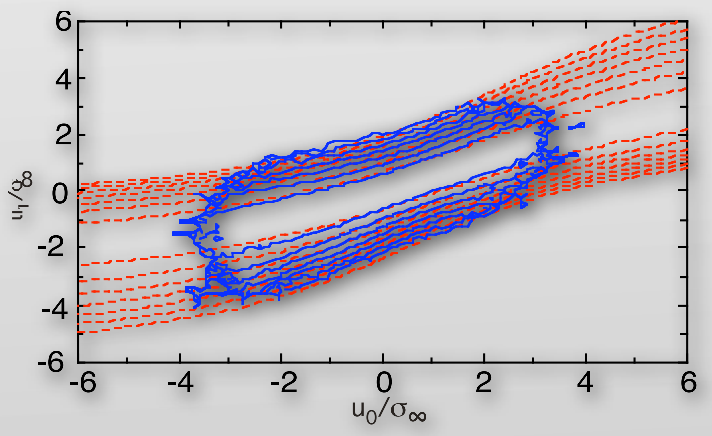
stochastic cascade process -6-

3. Verification of the measured Fokker-Planck equation

- numerical solution compared with experimental results
- => n-scale statistics



$$p(u_r, r | u_{r_0}, r_0)$$



Journal of Fluid Mechanics 433 (2001)
Phys. Rev E 76, 056102 (2007)

stochastic cascade process

Kolmogorov Obukhov 41:

$$\partial_r u_r = \frac{1}{3}\frac{u_r}{r}$$

Kolmogorov Obukhov 62

$$\partial_r u_r = \gamma\frac{u_r}{r} + \sqrt{Q\frac{u_r^2}{r}}\eta(r);$$

$$\gamma = 2Q - 1/3;\ Q = \frac{\mu}{18}$$

PRL 78 (1997)

PHYSICAL REVIEW E **71**, 027101 (2005)

Langevin equations from time series

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A. Porporato*
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after Pope and Ching

S. B. Pope and E. S. C. Ching, Phys. Fluids A **5**, 1529 (1993).

$$p(x) = \frac{N'}{\langle\langle \dot{x}^2 | x \rangle\rangle} \exp\left[\int_x \frac{\langle\langle \ddot{x} | u \rangle\rangle}{\langle\langle \dot{x}^2 | u \rangle\rangle} du \right],$$

Stat. Solution of Fokker Planck

$$\frac{\partial p(x,t)}{\partial t} = -\frac{\partial}{\partial x}[A(x)p(x,t)] + \frac{1}{2}\frac{\partial^2}{\partial x^2}[B(x)p(x,t)],$$

$$p(x) = \frac{N}{B(x)} \exp\left[2\int_x \frac{A(u)}{B(u)} du \right],$$

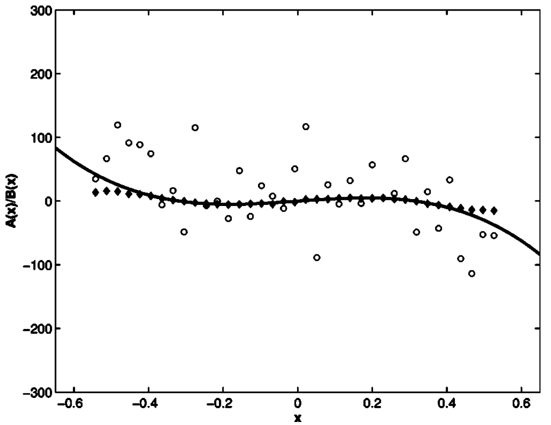


FIG. 3. Comparison among the estimated values of the ratio $\langle\langle \ddot{x}|x\rangle\rangle / \langle\langle \dot{x}^2|x\rangle\rangle$ using Sokolov's formulas (open circles), the ratio of the estimated drift and diffusion terms, $\langle \Delta x\rangle / \langle \Delta x^2\rangle$, using Eqs. (4) and (5) (solid diamonds), and the theoretical value, $A(x)/B(x)$ (solid line), for the pitchfork bifurcation process.

complexity of turbulence

thermodynamical (nonequilibrium) interpretation

- the Fokker- Planck or Kolmogov equation gives access

ideal gas	isotropic turbulence
state vector $\vec{q} = \begin{pmatrix} \vec{x} \\ \vec{p} \end{pmatrix}$	state vector u_r,
n- particle description $p(q_1, q_2, \ldots, q_n)$	n- scale statistics $p(u_{r0}, u_{r1}, \ldots, u_{rn})$
single particle approximation $p(q_1, \ldots, q_n) = p(q_1) * \ldots * p(q_n)$	Markov property $p(u_{r0}, \ldots, u_{rn}) = p(u_{r0} \vert u_{r1}) * \ldots * p(u_{rn-1} \vert u_{rn})\, p(u_{rn})$
Boltzmann equation $\partial_t p(q_i) = \ldots$	Fokker-Planck equation $-r\partial_r p(u_r \vert u_{r0}) = L_{FP}\, p(u_r \vert u_{r0})$

turbulence: new insights

Einstein- Markov-length - a coherence length

statistics of longitudinal and transversal increments

universality of turbulence:

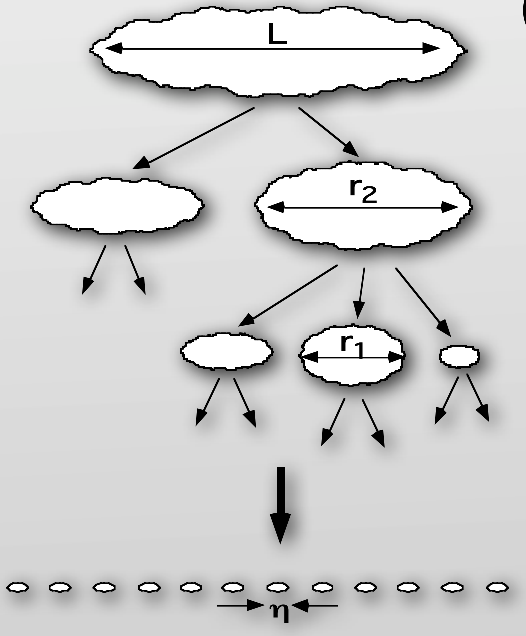
role of transfered energy e_r:

fusion rules $r_i \;=>\; r_{i+1}$ (Davoudi, Tabar 2000; L'vov, Procaccia 1996)

passive scalar (Tutkun, Mydlarski 2004)

turbulent length scales

turbulent cascade: larger Re larger cascade range



L integral length scale

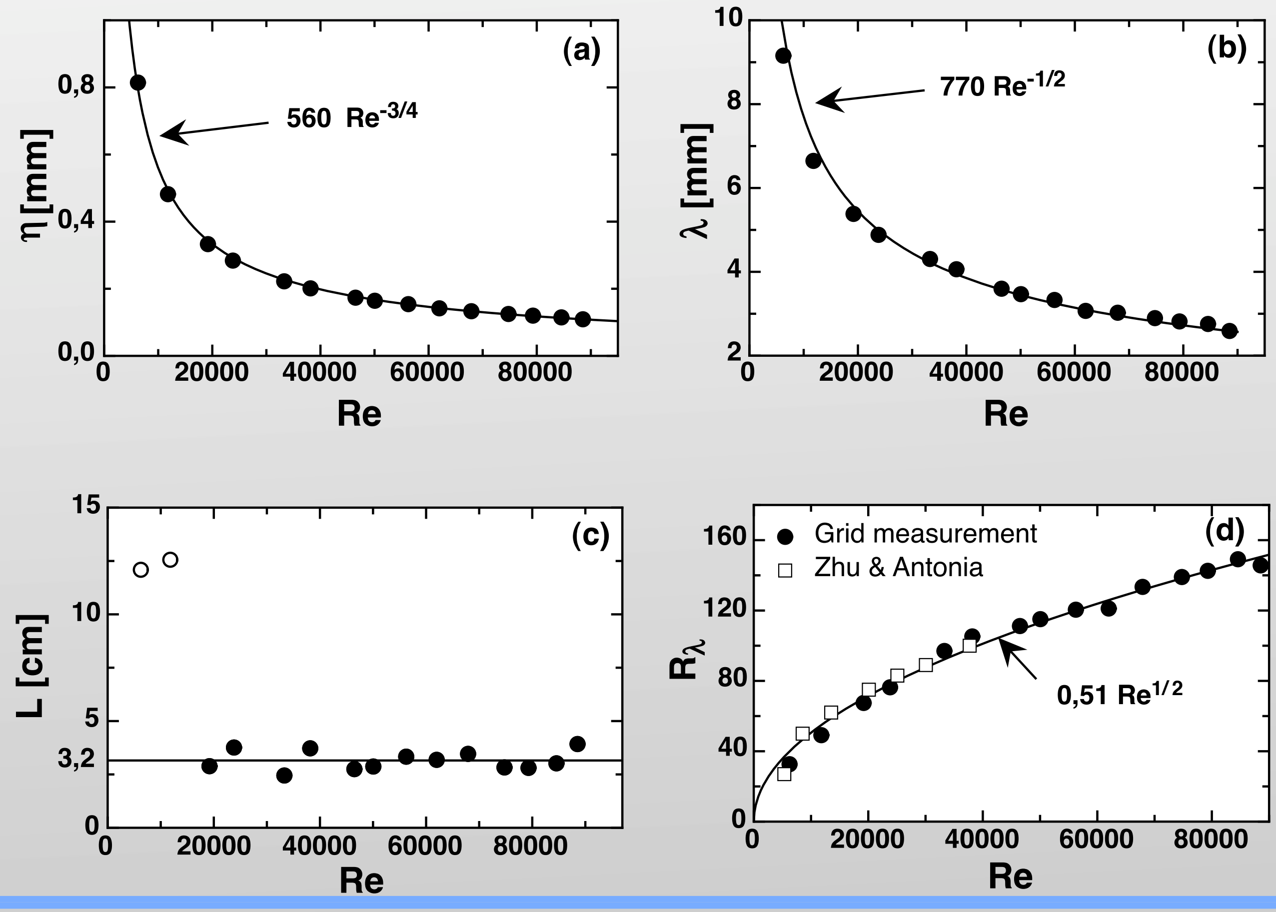
$$Re = \left(\frac{L}{\eta}\right)^{3/4}$$

λ Taylor length scale

η dissipation length scale

turbulent length scales

from grid experiments

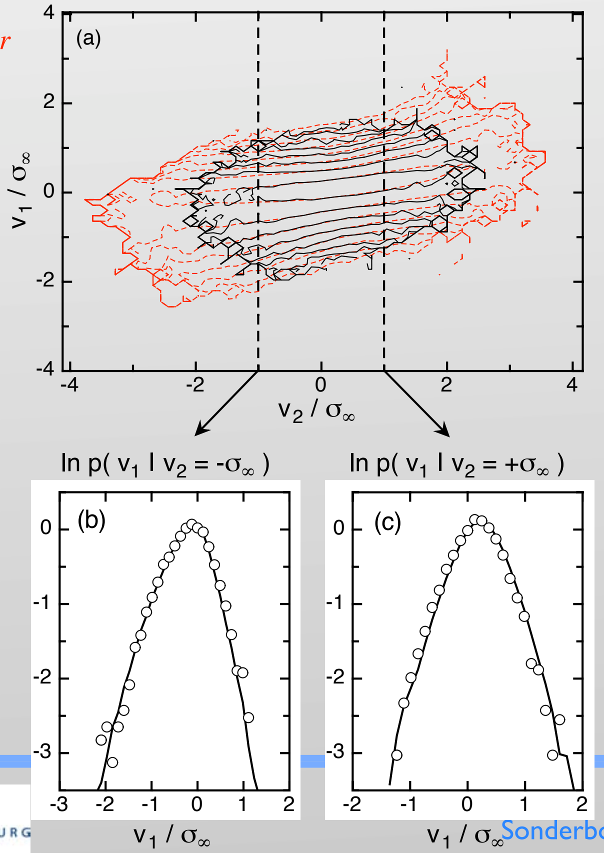


Einstein-Markov length

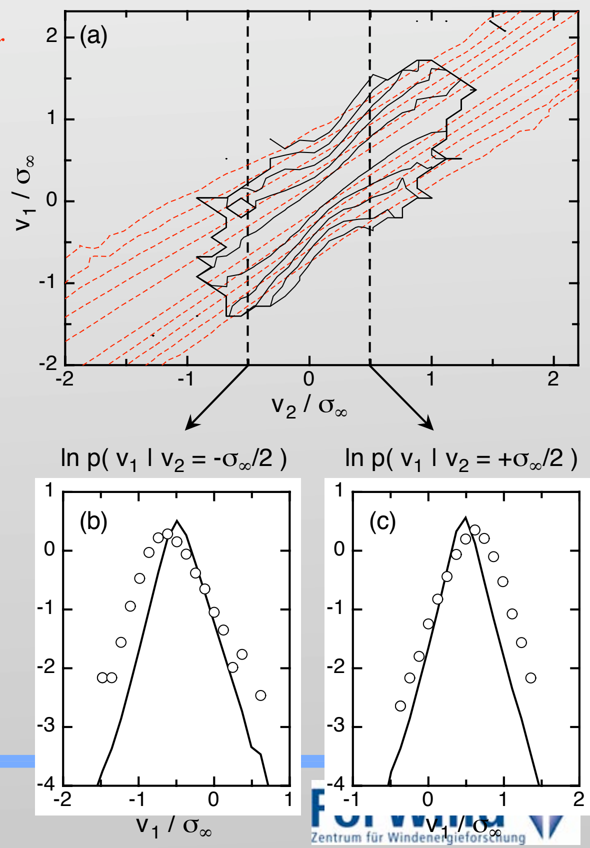
Einstein-Markov-length - a coherence length l_{mar}

$$p(\vec{u}_1, r_1 | \vec{u}_2, r_2; \ldots; \vec{u}_n, r_n) = p(\vec{u}_1, r_1 | \vec{u}_2, r_2)$$

$r_2 - r_1 > l_{mar}$

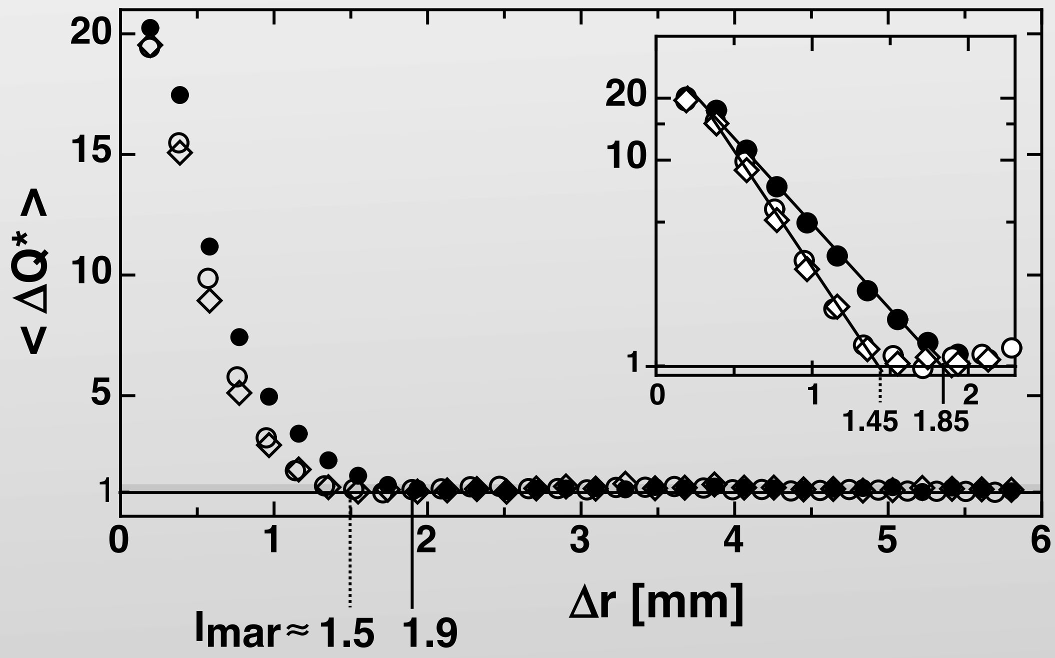


$r_2 - r_1 < l_{mar}$



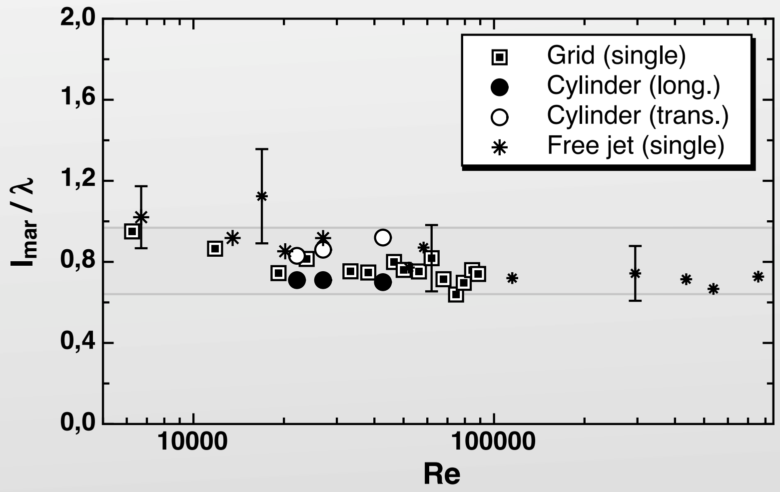
Einstein-Markov length -2-

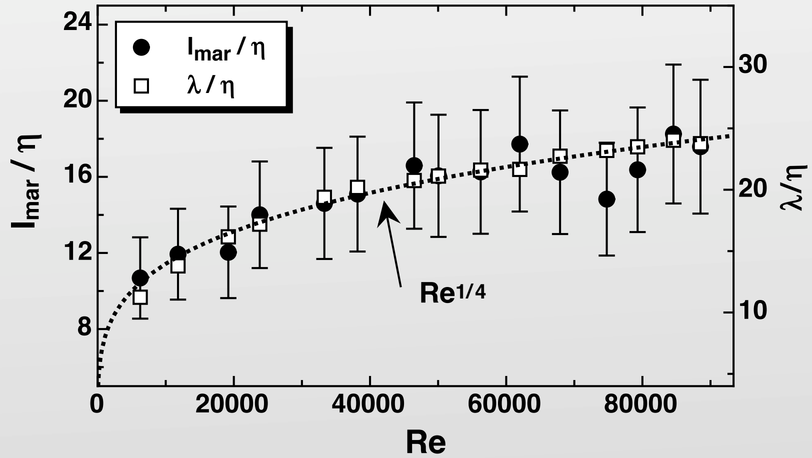
stochastic Wilcoxon test defines l_{mar}



Einstein-Markov length -3-

Einstein-Markov length l_{mar} a new coherence length





- is about the Taylor length
- is like the maximal dissipation length proposed by Yakhot
- dissipation causes memory
- degree of freedom L/l_{mar} like $Re^{1/2}$

Markov-Einstein Length

A. Einstein Ann. Phys. 17, 549 (1905)

5. ***Über die von der molekularkinetischen Theorie der Wärme geforderte Bewegung von in ruhenden Flüssigkeiten suspendierten Teilchen; von A. Einstein.***

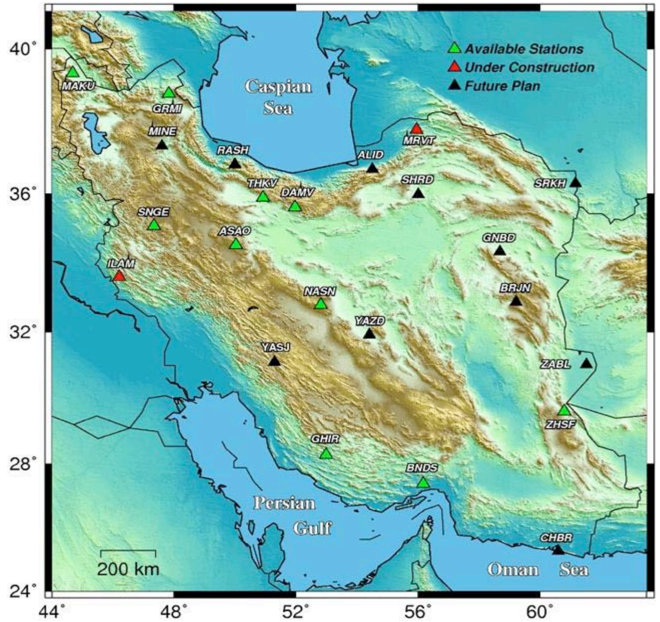
§ 4. Über die ungeordnete Bewegung von in einer Flüssigkeit suspendierten Teilchen und deren Beziehung zur Diffusion.

Wir gehen nun dazu über, die ungeordneten Bewegungen genauer zu untersuchen, welche, von der Molekularbewegung der Wärme hervorgerufen, Anlaß zu der im letzten Paragraphen untersuchten Diffusion geben.

Es muß offenbar angenommen werden, daß jedes einzelne Teilchen eine Bewegung ausführe, welche unabhängig ist von der Bewegung aller anderen Teilchen; es werden auch die Bewegungen eines und desselben Teilchens in verschiedenen Zeitintervallen als voneinander unabhängige Vorgänge aufzufassen sein, solange wir diese Zeitintervalle nicht zu klein gewählt denken.

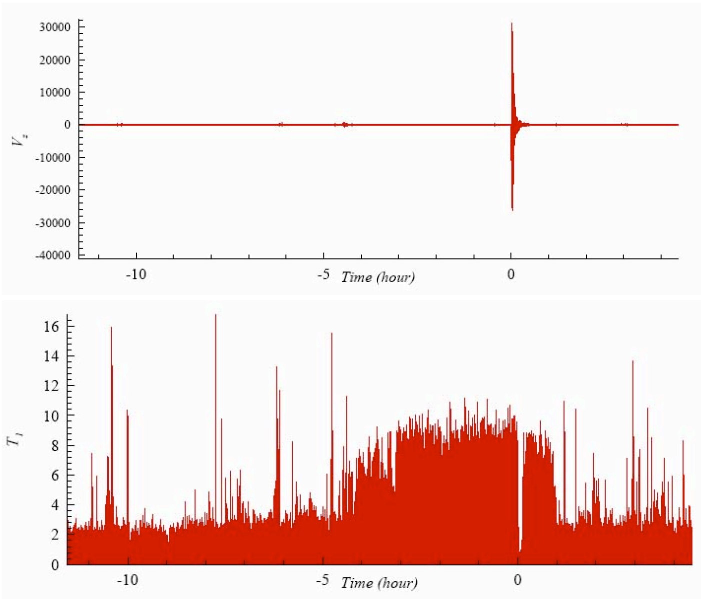
Wir führen ein Zeitintervall τ in die Betrachtung ein, welches sehr klein sei gegen die beobachtbaren Zeitintervalle, aber doch so groß, daß die in zwei aufeinanderfolgenden Zeitintervallen τ von einem Teilchen ausgeführten Bewegungen als voneinander unabhängige Ereignisse aufzufassen sind.

Einstein-Markov length - for seismic data

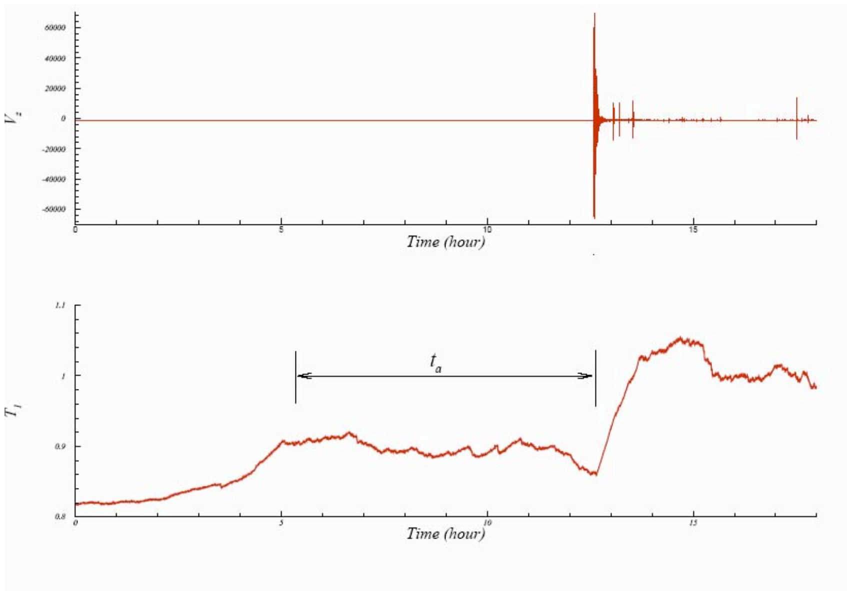


Einstein-Markov length - for seismic data

Saravan, 13/03/2005, Ms=5.4



Baladeh, Iran, 28/05/2004, M=6.4



M.R.R. Tabar, et.al. Lecture Notes in Physics , Vol. 705, (Springer, 2006) 281-301.

turbulence: further results

spatial correlation in different directions

Quantities

- longitudinal increment

$$u_r(x) = \left[\vec{u}(\vec{x}+\vec{r}) - \vec{u}(\vec{x})\right] \cdot \hat{r}$$

- transversal increment

$$v_r(x) = \left|\left[\vec{u}(\vec{x}+\vec{r}) - \vec{u}(\vec{x})\right] \times \hat{r}\right|$$

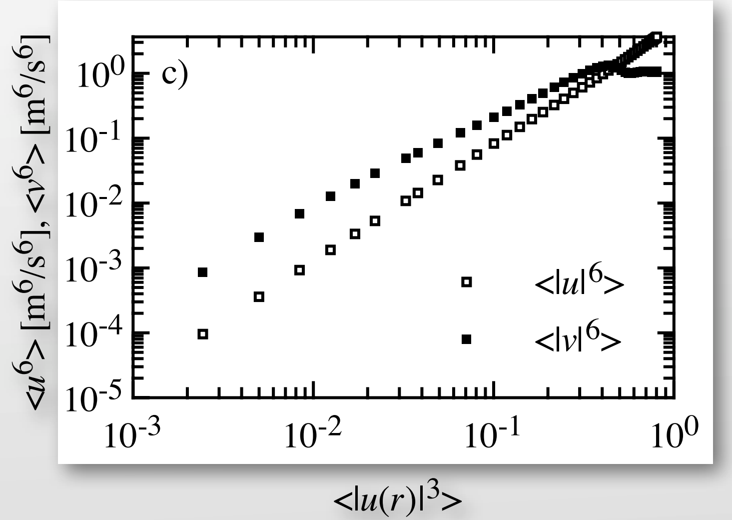
turbulence: long/transversal -2-

extended selfsimilartiy ESS

supposed scaling laws

$$\langle |u_r|^n \rangle \propto \langle |u_r|^3 \rangle^{\xi_n^l}$$

$$\langle |v_r|^n \rangle \propto \langle |u_r|^3 \rangle^{\xi_n^t}$$



turbulence: long/transversal -2-

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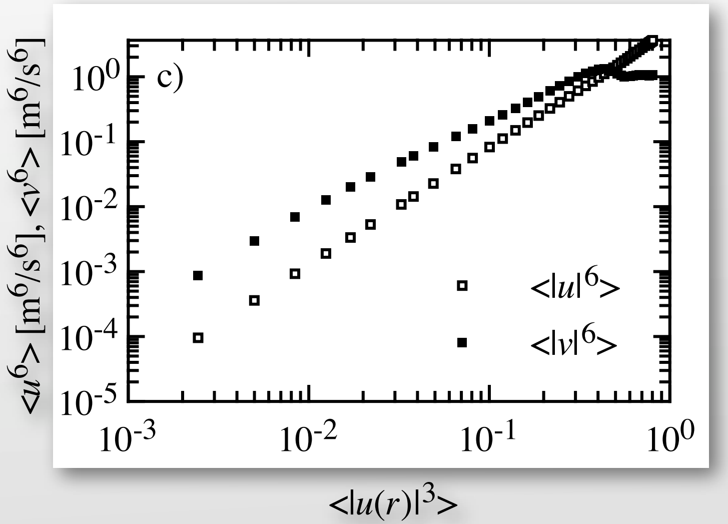
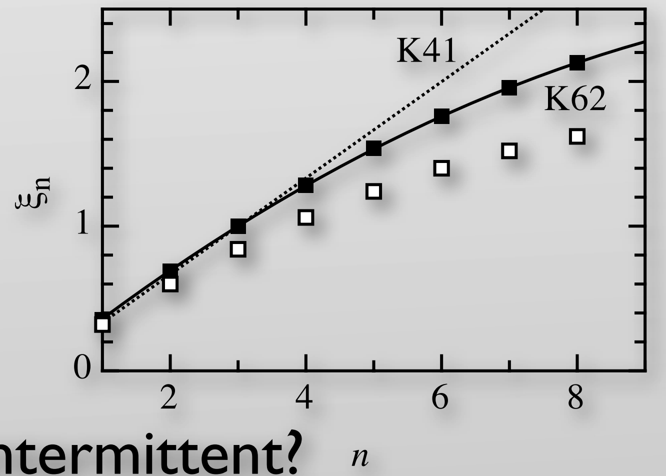


open problem: (Antonia 97, Benzi 97, van der Water 99, Grossman et.al. 97....)



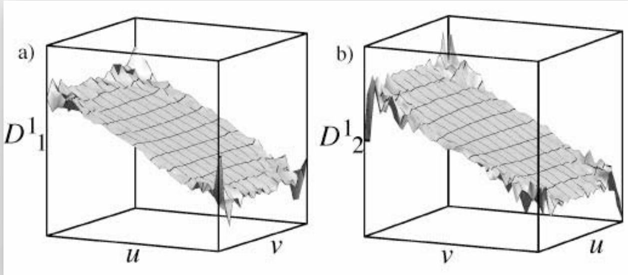
$$\xi_n^t > \xi_n^l$$

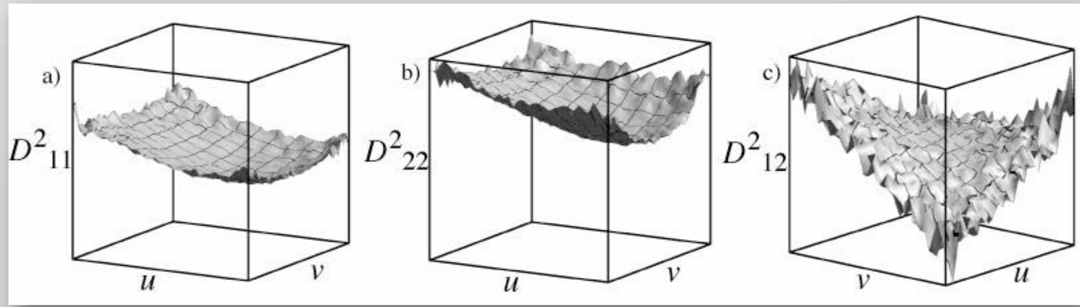
are transversal structures more intermittent?

turbulence: long/transversal -3-

$$-r\frac{\partial}{\partial r}p(\mathbf{u},r|\mathbf{u_0},r_0) =$$

$$\left(-\sum_{i=1}^{n}\frac{\partial}{\partial u_i}D_i^{(1)} + \sum_{i,j=1}^{n}\frac{\partial^2}{\partial u_i \partial u_j}D_{ij}^{(2)}\right)p(\mathbf{u},r|\mathbf{u_0},r_0)$$





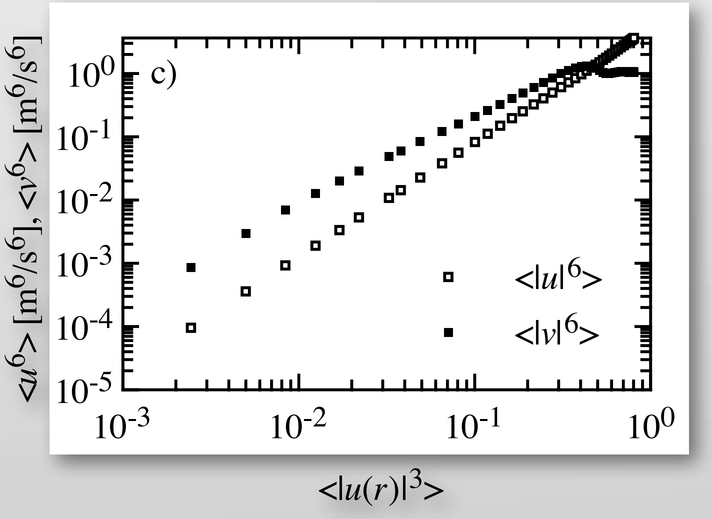
turbulence: long/transversal -4-

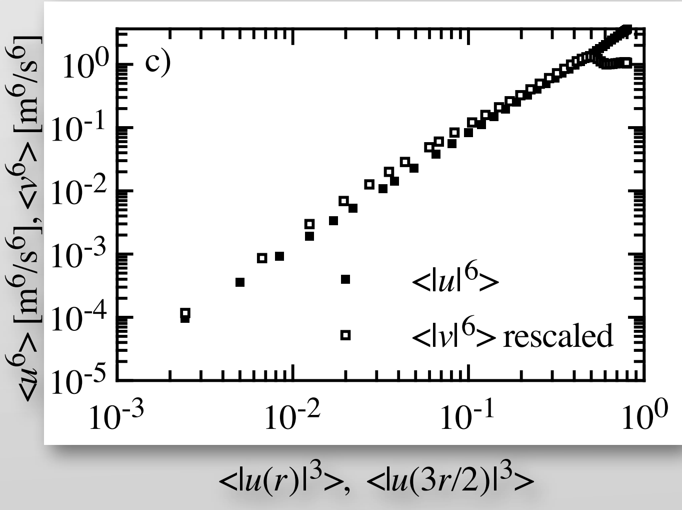
rescaling symmetry: *r => 3r/2*

$$< |v(r)|^n > \propto < |u(r)|^3 >^{\xi_n^t}$$

new ESST :

$$< |v(r)|^n > \propto < |u(3r/2)|^3 >^{\xi_n}$$





turbulence: long/transversal -4-

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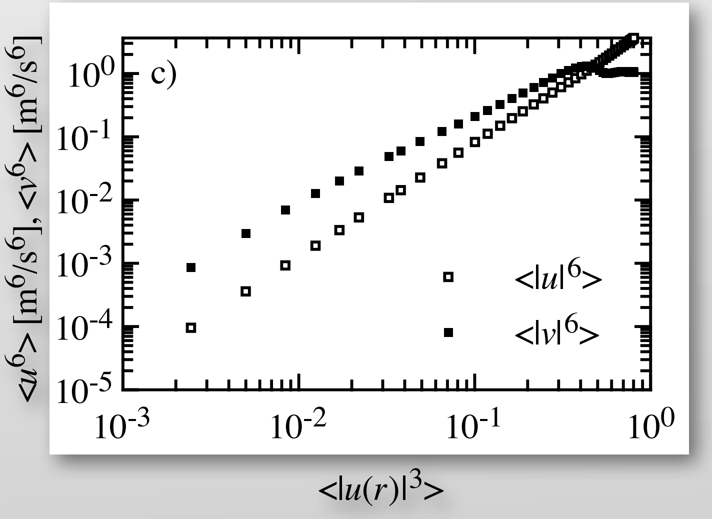
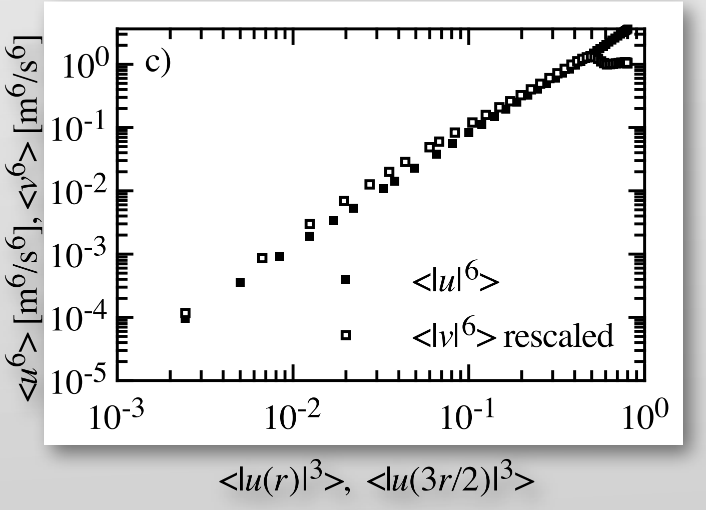




striking result - this is only possible if the scaling laws $< |v(r)|^n > \propto r^{\xi_n^t}$ does not hold

turbulence: long/transversal -4-

rescaling symmetry: *r => 3r/2*

$$< |v(r)|^n > \propto < |u(r)|^3 >^{\xi_n^t}$$

$$< |v(r)|^n > \propto < |u(3r/2)|^3 >^{\xi_n}$$

consitent with Karman equation:

$$-r\frac{\partial}{\partial r}\langle u_r^{\ 2}\rangle = 2\langle u_r^{\ 2}\rangle - 2\langle v_r^{\ 2}\rangle$$

or

$$\langle v_r^{\ 2}\rangle = \langle u_r^{\ 2}\rangle + \frac{r}{2}\frac{\partial}{\partial r}\langle u_r^{\ 2}\rangle$$

taken as Taylor series $\langle v_r^{\ 2}\rangle \approx \langle u_{3/2r}^{\ \ 2}\rangle$

Phys. Rev. E 70(5) (2004)

turbulence: long/transversal -5-

universality of turbulence:

$D^{(1)}(u,r) \cong \gamma(r)\, u(r)$

$D^{(2)}(u,r) \cong \alpha(r,Re) + \delta(r)\, u(r) + \beta(r,Re)\, u^2(r)$

=> Exp: cascade process depends on Re

Phys. Rev. Lett. **89**, (2002)

roll of transfered/dissipated energy e_r:,

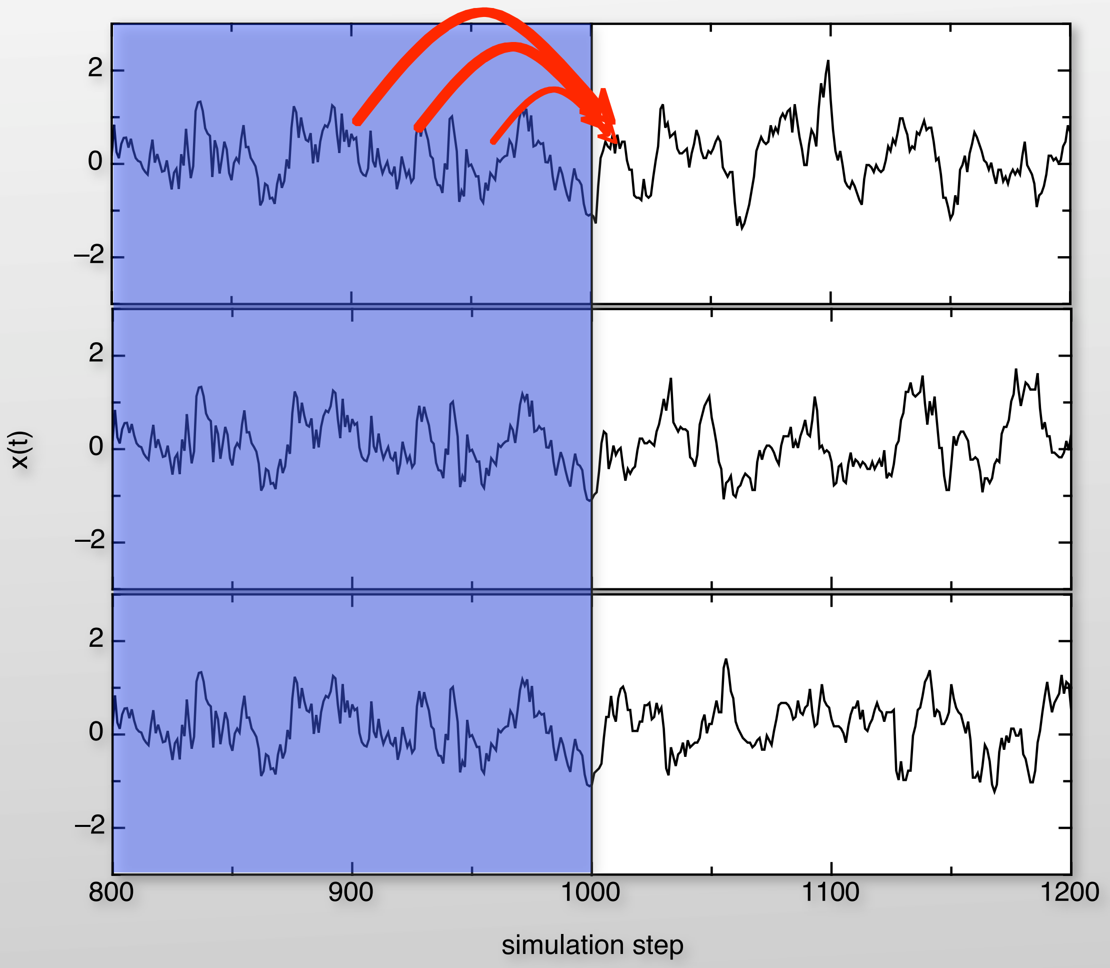
$D^{(2)}(u,r,\, e_r) \cong \alpha(r) + m\, f(e_r)$

$D^{(2)}$ does not any more lead to multiplicative noise

=> e_r causes intermittency of the velocity field

Renner et.al in preparation
see also Gagne st al 1994;
Naert et al 1998

reconstruction of time series



use of increments alined to the right

$$p(u(x_{new}) \mid u(x_1) \ldots, u(x_n))$$

is given by

$$p(u_1, r_1; u_2, r_2; \ldots; u_n, r_{n-1})$$

Nawroth et al Phys. Lett. (2006)

multiplier statistics

since Kolomogorov 62 idea of multipliers (for increments)

$$w_n := u_{n+1}/u_n$$

multiplier statistics

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$$w_n := u_{n+1}/u_n$$

$$p(w_n) = \int \delta\left(w_n - \frac{u_{n+1}}{u_n}\right) p(u_{n+1}, u_n) du_{n+1} du_n.$$

Fokker-Planck equ. with
$$D^{(1)}(u,r) = \gamma(r)u$$
$$D^{(2)}(u,r) = \alpha(r)$$

Chauchy distribution with parameters
λ and b given by $D^{(1)}$ and by $D^{(2)}$

$$p(w_n) = \frac{1}{\pi} \frac{\lambda_{n+1}}{\lambda_{n+1}^2 + (b_{n+1} - w_n)^2}$$

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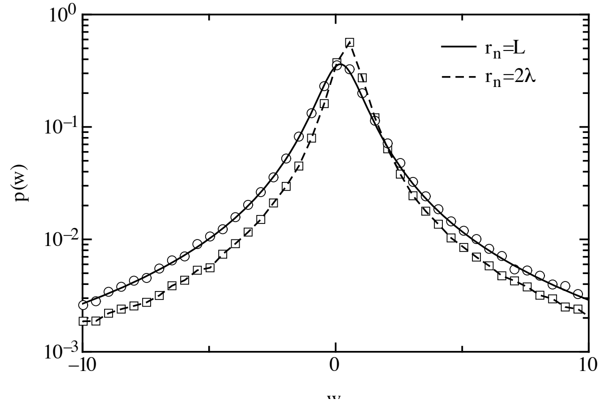
Chauchy distribution

arises if one divides two Gaussian stoch. variables

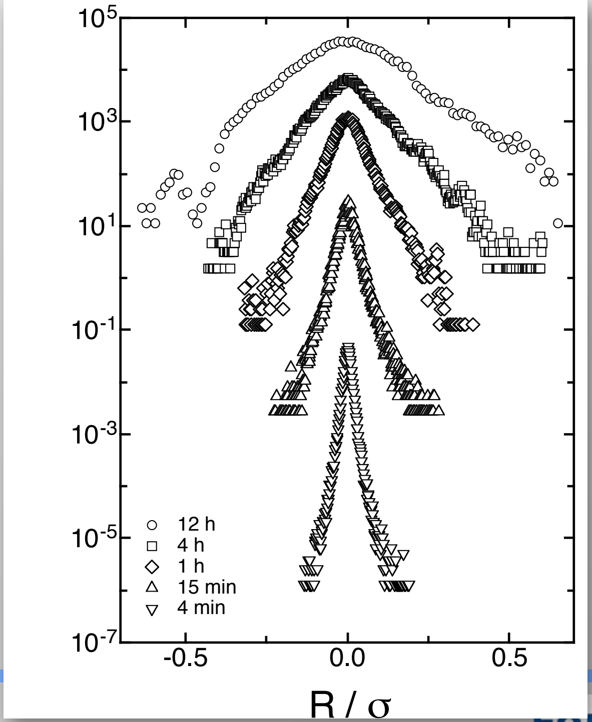


finance

scale dependent quantity for measuring the disorder

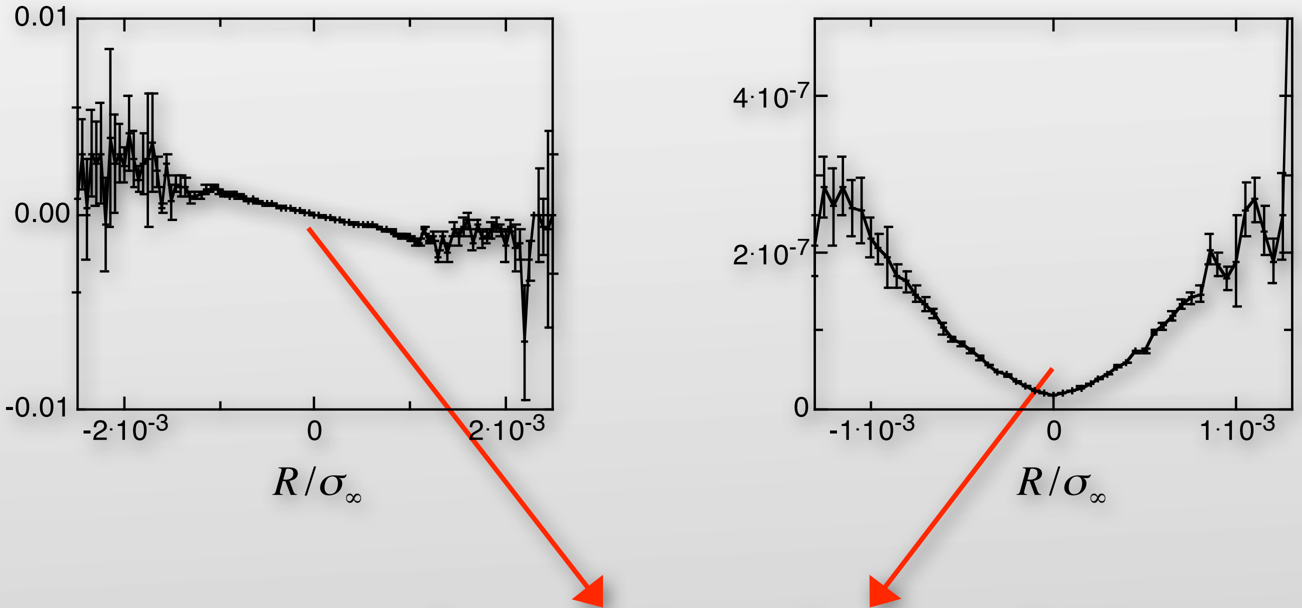
return or log return for different time scales

$$Q(x,r) \Rightarrow r(t,\tau) = \frac{x(t+\tau)}{x(t)} \quad \text{or } R(t,\tau) = \log r(t,\tau)$$



finance -2-

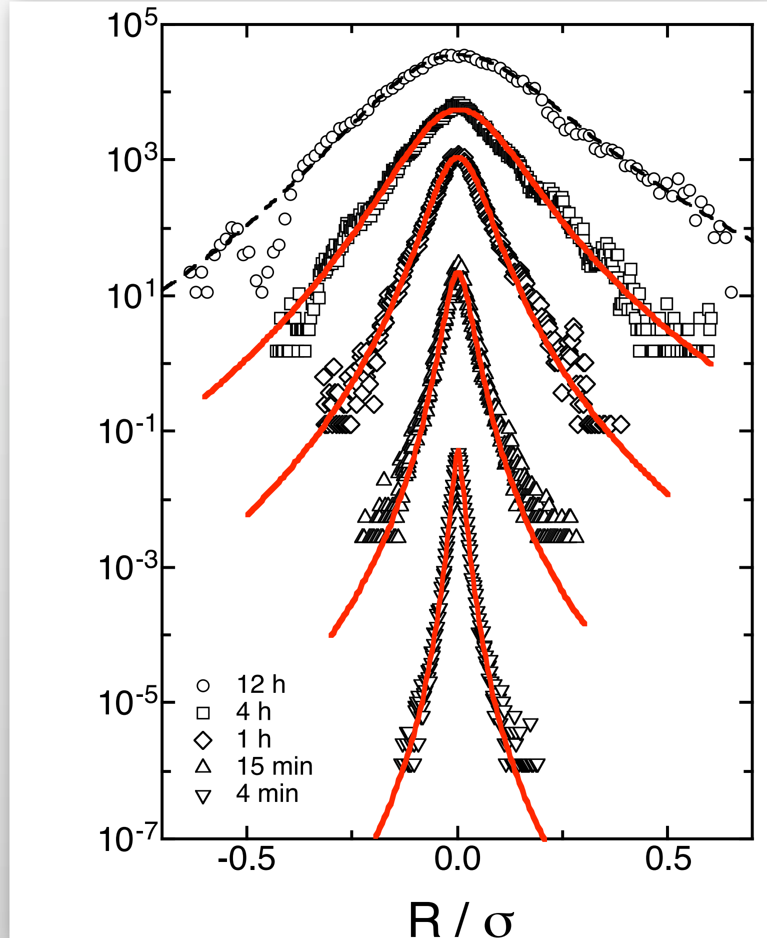
Functional form of the coefficients $D^{(1)}$ and $D^{(2)}$ is presented



$$\frac{\partial}{\partial \tau} p(R,\tau) = \left[-\frac{\partial}{\partial R} D^{(1)}(R,\tau) + \frac{\partial^2}{\partial R^2} D^{(2)}(R,\tau) \right] p(R,\tau)$$

Example: Volkswagen, τ = 10 min

finance -3-



comparison of data with numerical solution of the Kolmogorov equation

Physica A **298** ,499 (2001)

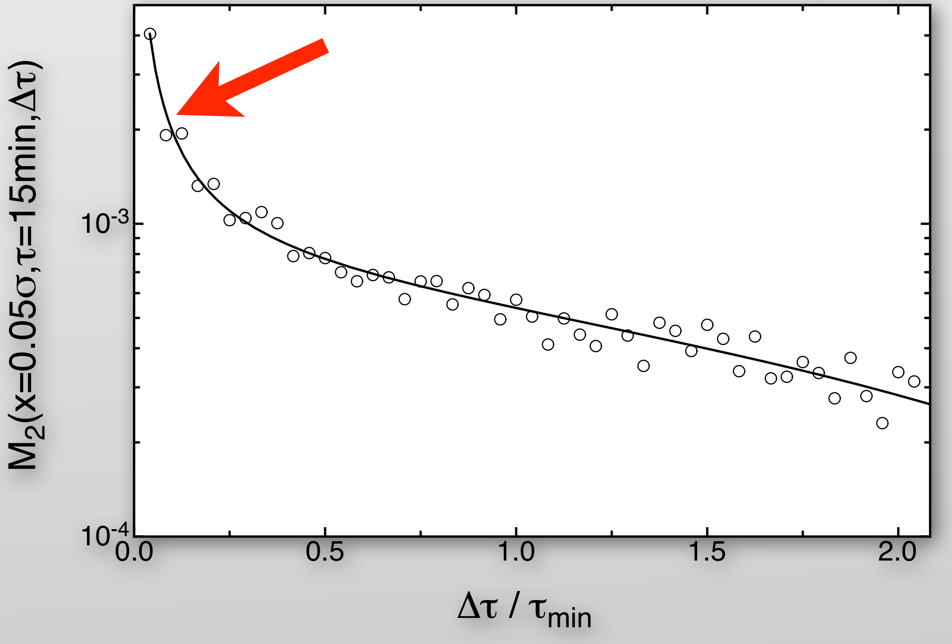
Does the method always work ?

further applications for time series

finance

the estimation of the Kramers Moyal coefficient gives divergencies for $\Delta\tau \to 0$

$$D^{(n)}(R) = \lim_{\Delta\tau\to 0} \frac{1}{n!\cdot\Delta\tau} \int (R' - R)^n \, p(R',\tau + \Delta\tau \mid R,\tau) dR'$$



FX DM/$ Olsen data

(Physica A **298**, 499 (2001))

finance

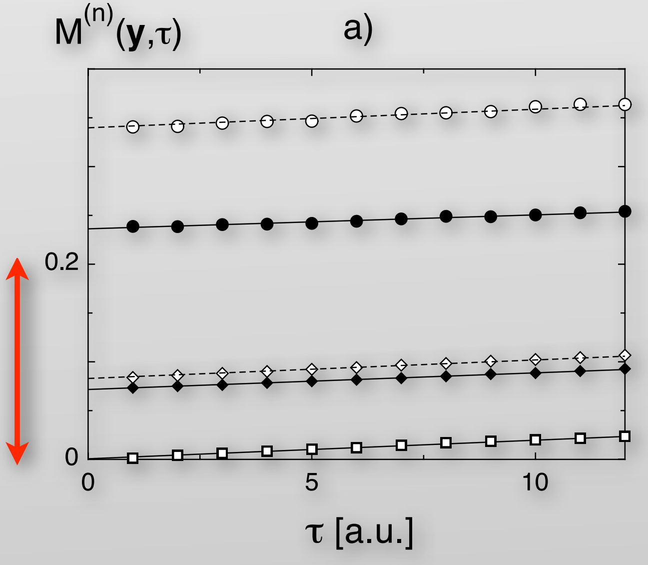
divergent Kramers Moyal coefficients are due to measurement noise (jump processes)

(Physica A **298**, 499 (2001), Euro. Phys. Lett.61(2003); F. Böttcher, D. Kleinhans Phys. Rev. Lett. 97 (2006)

process variable $x(t) => y(t) = x(t) + \sigma \cdot \eta(t)$

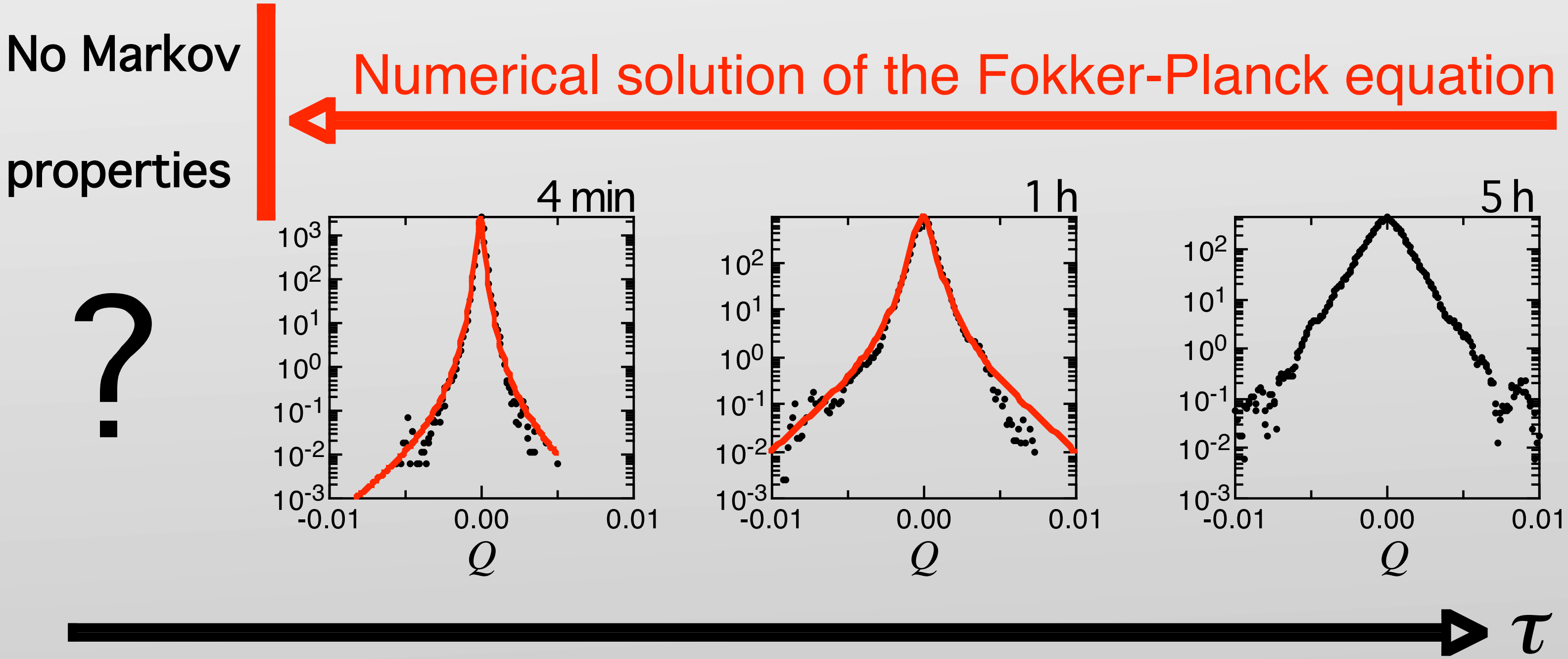
$$\tau \cdot M^{(1)}(y,\tau) = \tau \cdot D^{(1)}(x = y) + \gamma_1(y,\sigma)$$

$\gamma_1(y,\sigma)$



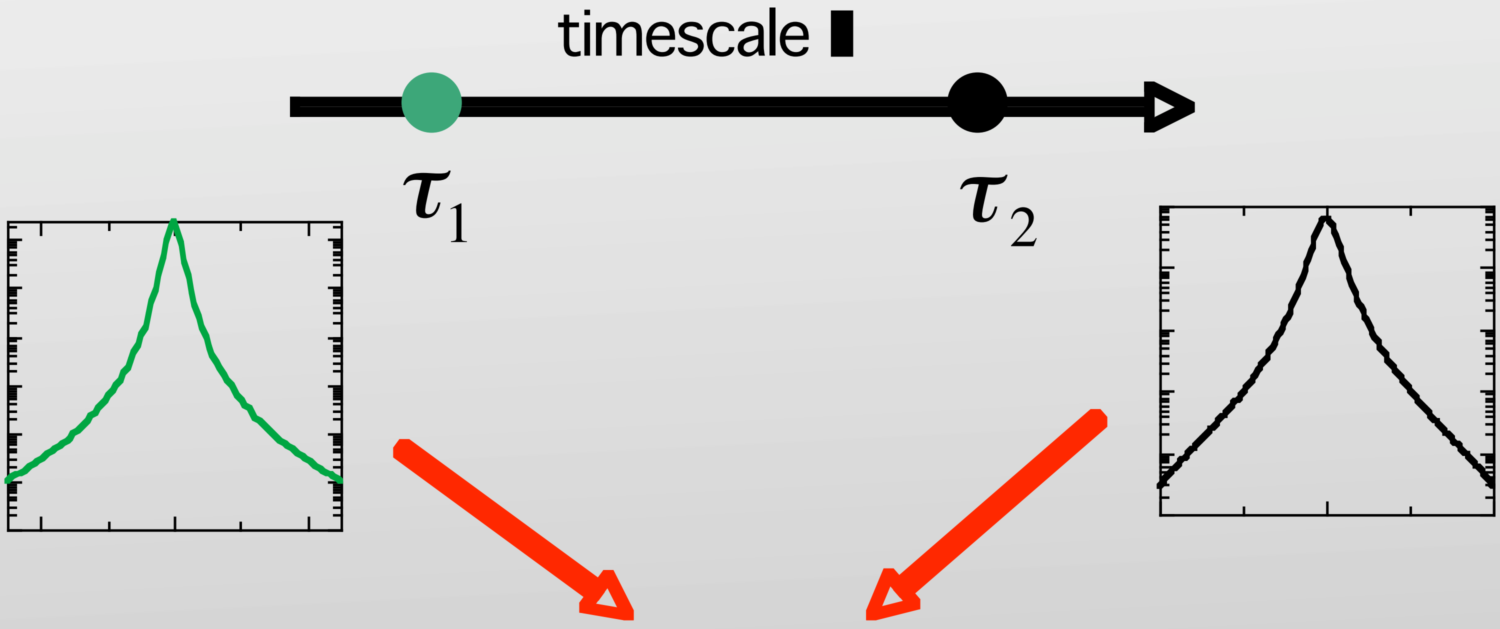
universal small scale statistics

Numerical solution of the Fokker-Planck equation for the coefficients $D^{(1)}$ and $D^{(2)}$, which were directly obtained from the data.



universal small scale statistics

The reference distribution & The considered distribution



measure of distance d

Comparison of p_N and p_R - The Measures

-- Kullback-Leiber-Entropy:

$$d_K(p_N(Q,\tau),p_R) \equiv \int_{-\infty}^{+\infty} p_N(Q,\tau)\cdot \ln\left(\frac{p_N(Q,\tau)}{p_R}\right)\cdot dQ$$

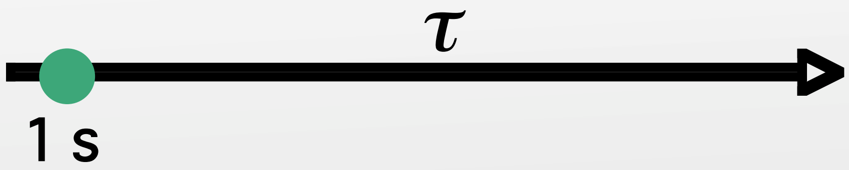
-- Weighted mean square error in logarithmic space:

$$d_M(p_N(Q,\tau),p_R) \equiv \frac{\int_{-\infty}^{+\infty} (p_R + p_N(Q,\tau))\cdot\left(\ln(p_N(Q,\tau)) - \ln(p_R)\right)^2 \cdot dQ}{\int_{-\infty}^{+\infty} (p_R + p_N(Q,\tau))\cdot\left(\ln^2(p_N(Q,\tau)) + \ln^2(p_R)\right)\cdot dQ}$$

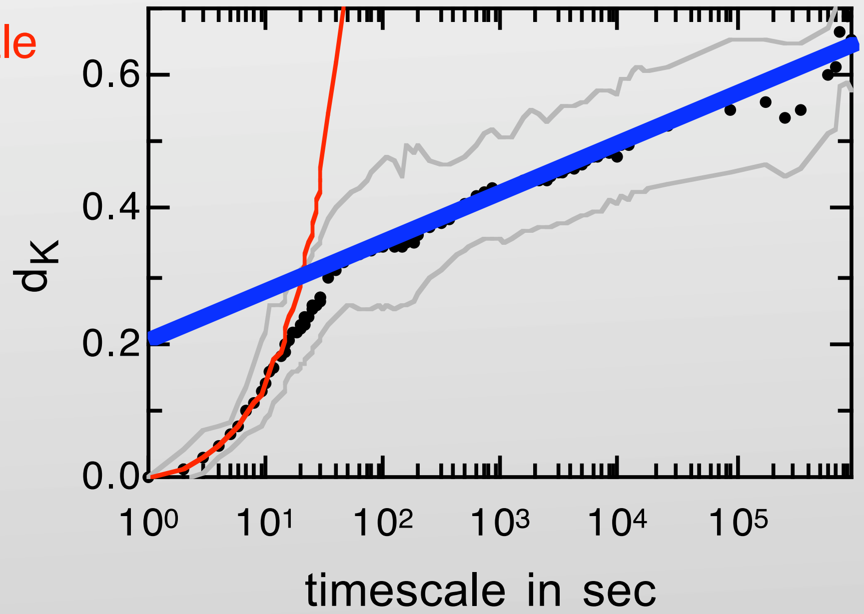
-- Chi-square distance:

$$d_C(p_N(Q,\tau),p_R) \equiv \frac{\int_{-\infty}^{+\infty} \left(p_N(Q,\tau) - p_R\right)^2 \cdot dQ}{\int_{-\infty}^{+\infty} p_R \cdot dQ}$$

universal small scale statistics



Small Timescale Regime
Non Markov

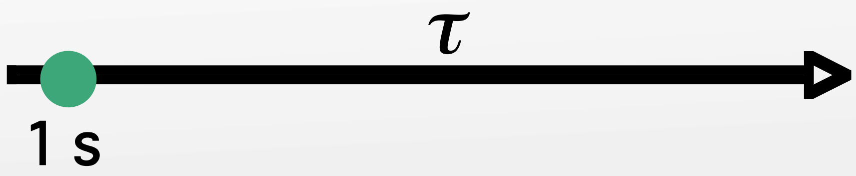


Fokker-Planck Regime
Markov process

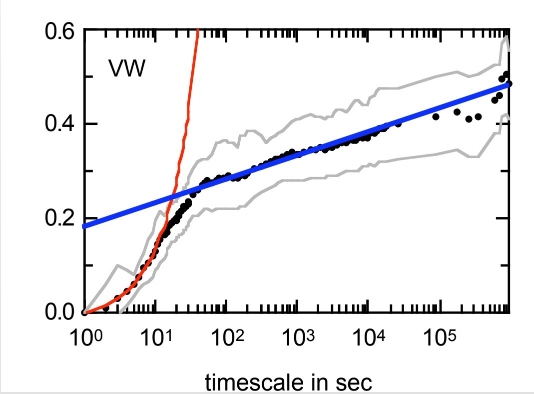
Small timescales are special !

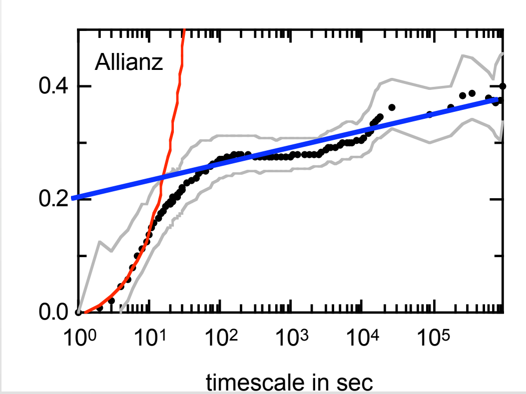
Example: Volkswagen

universal small scale statistics

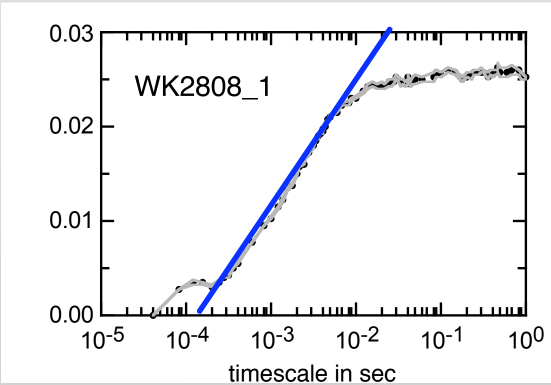


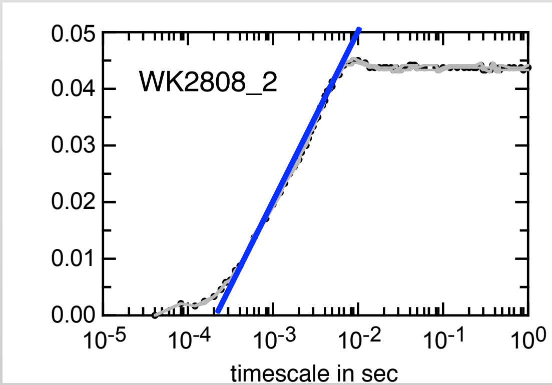
finance





turbulence





Eur. Phys. J. B 50, 147–151 (2006)

universal small scale statistics

scale dependent complexity

scale

τ

subrange | inertia range

Non Markov

Finance

New Universal Feature?

Small Timescale Regime

Turbulence

viscous subrange

Markov process

- Fokker-Planck Regime -

Physica A **382**, 193 (2007)

$$dX_t = b(X_t, t)dt + \sigma(X_t, t)dw_t$$

$$b(X_t, t) = D^{(1)}(X_t, t)$$
$$\sigma^2(X_t, t) = D^{(2)}(X_t, t)$$

END

shown that for stochastic processes drift and diffusion can be measured see also http://www.physik.uni-oldenburg.de/hydro/20660.html

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