On the design and analysis of branching schemes with killing for rare event Monte Carlo estimation

Paul Dupuis

(with Thomas Dean, Oxford University)
Division of Applied Mathematics
Brown University

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3. Representations and bounds for the second moment
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6. Remarks

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Assume $\{X^n(\cdot)\}$ satisfies a Large Deviation Principle with rate function

$$I_T(\phi) = \int_0^T L(\phi, \dot{\phi}) dt$$

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if ϕ is AC and $I_T(\phi) = \infty$ else. Heuristically, for $T < \infty$, given ϕ, small $\delta > 0$, $x_n \to x = \phi(0)$ and large n

$$P_{x_n}\left\{\sup_{0\leq t\leq T} \|X^n(t) - \phi(t)\| \leq \delta\right\} \approx e^{-nI_T(\phi)}.$$

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To estimate:

$$E_{x_n}\left[\sum_{i=0}^{\tau^n} e^{-nF(X_i^n)}\right], \text{ where } \tau^n \doteq \inf\left\{i : X_i^n \in M\right\}.$$

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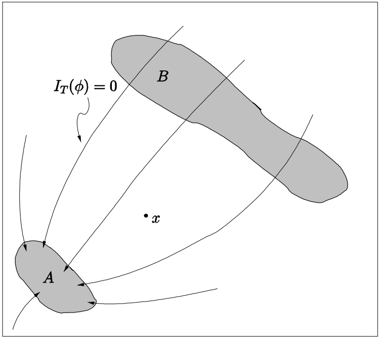
Example:

$M = A \cup B$, B rare, A typical, and $F(x) = 0$, $x \in B$, $F(x) = \infty$ otherwise.

Problem of Interest and LD Scaling



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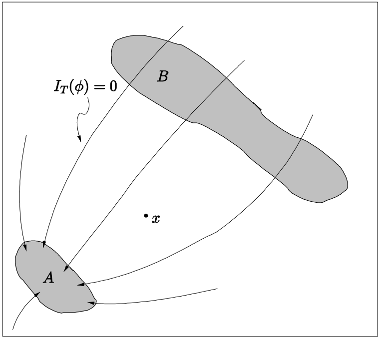

Under conditions of regularity on F and bounds on τ^n:

$$
\begin{aligned}
-\frac{1}{n}\log E_{x_n}\left[\sum_{i=0}^{\tau^n} e^{-nF(X_i^n)}\right] &\rightarrow \inf\left\{I_T(\phi) + F(\phi(T)) : \phi(0) = x, T < \infty\right\} \\
&= \gamma.
\end{aligned}
$$

Some Estimation Generalities

1. General approach: construct iid random variables $\theta_1^n, \ldots, \theta_K^n$ with $E\theta_1^n = E_{x_n}\left[\sum_{i=0}^{\tau^n} e^{-nF(X_i^n)}\right]$ and use the unbiased estimator

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$$-\frac{1}{n}\log E\left(\theta_1^n\right)^2 \leq -\frac{2}{n}\log E\theta_1^n = -\frac{2}{n}\log E_{x_n}\left[\sum_{i=0}^{\tau^n} e^{-nF(X_i^n)}\right] \to 2\gamma.$$

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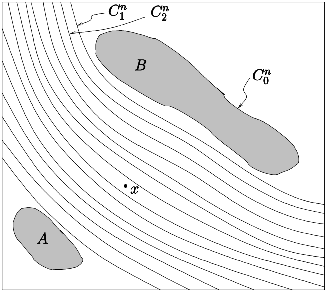
4. An estimator is called *asymptotically efficient* if
$$\liminf_{n\to\infty} -\frac{1}{n}\log E\left(\theta_1^n\right)^2 \geq 2\gamma.$$

Splitting-Type Schemes

A certain number [proportional to n] of *splitting thresholds* C_r^n are defined which enhance migration, e.g., for hitting probabilities:



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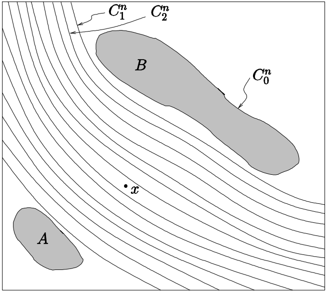

A single particle is started at x that follows the same law as X^n, but branches into a number of independent copies each time a new level is reached.

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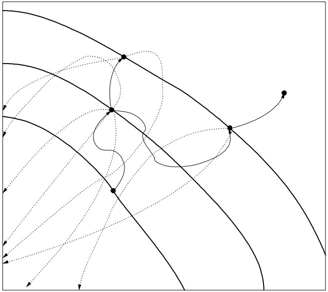

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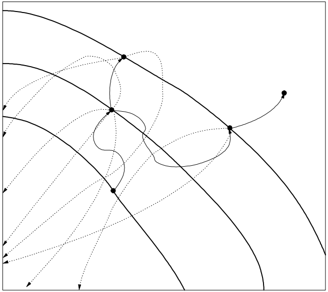

The number of new particles S can be random (though independent of past data), and a multiplicative weight w_i is assigned to the ith descendent, where

$$E\sum_{i=1}^{S} w_i = 1.$$

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Evolution continues until every particle hits M. Let

$$\begin{aligned} N_x^n &= \text{number of particles generated} \\ X_i^n(j) &= \text{trajectory of } j\text{th particle,} \\ W_i^n(j) &= \text{product of weights assigned to } j \text{ along path up to time } i \\ \tau^n(j) &= \text{hitting time of } j\text{th trajectory} \end{aligned}$$

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- If too frequent, we have exponential growth in number of surviving particles.

RESTART and DPR

An obvious inefficiency–continuing trajectories far from the places of interest. When killing trajectories, care needed to avoid bias.

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- Same problems as with ordinary splitting, but analysis much more difficult due to dependence on threshold of birth.

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and a mean increase in number of particles per threshold of

$$e^{n\Delta}.$$

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where $X_j^{k,i,n}$ are (conditionally) independent copies of X_j^n that start at X_i^n at $j = i$.

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Then

- U being a subsolution is a *necessary* and *sufficient* condition for subexponential growth in number of particles and total computational effort.
- If U is a subsolution

$$\lim_{n\to\infty} -\frac{1}{n}\log E\left[(\theta^n)^2\right] = \inf_y \left\{ \mathcal{J}(x,y) + (U(x) - U(y)) \vee 0 + 2F(y) \right\}$$

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Achieved, and hence asymptotic optimality, if at minimizing y

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The F-V Quasipotential and Subsolutions

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A special case. Product form or asymptotically product form stochastic networks, $Q(y) = \langle a, y \rangle$.

Example, Tandem Queue

$$U(y) = -\left[\log\left(\frac{\mu_1}{\lambda}\right)\right] y_1 - \left[\log\left(\frac{\mu_2}{\lambda}\right)\right] y_2,$$

$\lambda = 1, \mu_1 = \mu_2 = 4.5.$

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Shared buffer capacity n:

n	30	40	50
Theoretical Value	2.63×10^{-18}	1.03×10^{-24}	3.80×10^{-31}
Estimate	2.63×10^{-18}	1.06×10^{-24}	3.83×10^{-31}
Std. Err.	0.08×10^{-18}	0.04×10^{-24}	0.15×10^{-31}
95% C.I.	$[2.47, 2.79] \times 10^{-18}$	$[0.99, 1.14] \times 10^{-24}$	$[3.54, 4.13] \times 10^{-31}$
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Separate buffers each of capacity n:

n	10	20	30
Theoretical Value	9.64×10^{-8}	1.60×10^{-15}	2.64×10^{-23}
Estimate	9.70×10^{-8}	1.57×10^{-15}	2.64×10^{-23}
Std. Err.	0.16×10^{-8}	0.03×10^{-15}	0.06×10^{-23}
95% C.I.	$[9.39, 10.0] \times 10^{-8}$	$[1.51, 1.63] \times 10^{-15}$	$[2.53, 2.75] \times 10^{-23}$
Time Taken (s)	3	12	26

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- There is an analogous theory for importance sampling, but it imposes stronger conditions on the subsolution. Differences may be significant.