Modelling stochastic volatility with ambit processes

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Motivation

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1. Log-volatility is persistent/has long memory (Andersen et al., 2003-E).
2. Log-volatility is rough (Gatheral et al., 2014-WP).
3. Log-volatility is non-Gaussian (Lunde, Pakkanen, MB., 2016-WP).

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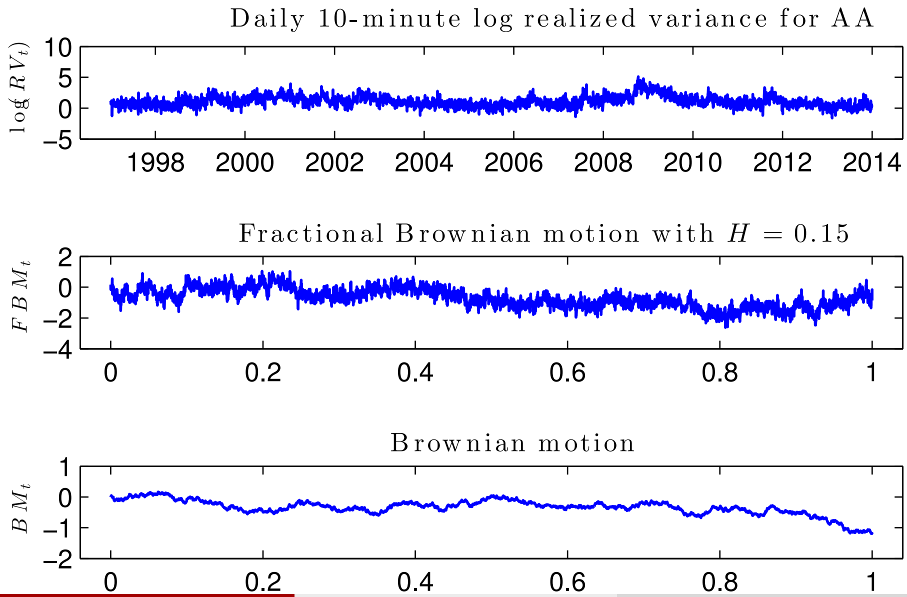
This talk:

1. Empirical evidence.
   - From high frequency data (E-mini S&P 500; 2013Q1-2014Q4).
   - From daily volatility data (2000+ assets; period: 2003Q1-2013Q4).
2. Modelling log-volatility by $\mathcal{BSS}$ processes.

Outline

1. Empirical evidence
   - Roughness of log-volatility
   - Persistence of log-volatility
2. Modelling log-volatility by $\mathcal{BSS}$ processes
3. Final remarks

A first look at rough volatility



Extracting volatility

- Volatility is latent. Let $\Delta > 0$ and define

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where $\widehat{IV}$ is an estimator of integrated variance.

- Many possibilities for $\widehat{IV}$ exist (realized variance, two-scale estimators, preaveraging estimators, realized kernels). We choose the realized kernel (Barndorff-Nielsen et al., 2008-E).

Outline

Defining roughness

Definition

Suppose that a process X is covariance stationary with ACF

$$\rho(h) := Corr(X_{t+h}, X_t), \quad h, t \in \mathbb{R}.$$

We say that X is rough if

$$1 - \rho(h) \sim C|h|^{2\alpha+1} L(h), \quad h \to 0,$$

for L slowly varying, a constant $C > 0$ and $\alpha \in (-1/2, 0)$.

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Remark

1. The Brownian motion has $\alpha = 0$.
2. An H-fBm has $\alpha = H - 1/2$.
3. In general, rough Gaussian processes has γ-Hölder continuous trajectories for all $\gamma \in (0, \alpha + 1/2)$.

Estimating roughness

Semiparametric estimator of α

We assume

$$1 - \rho(h) \sim C|h|^{2\alpha+1}, \quad h \to 0.$$

Then, estimate $\rho(h)$ from the data and run the regression:

$$\log(1 - \hat{\rho}(h)) = c + (2\alpha + 1)\log|h| + \epsilon_h, \quad h = \Delta, 2\Delta, \ldots, m\Delta$$

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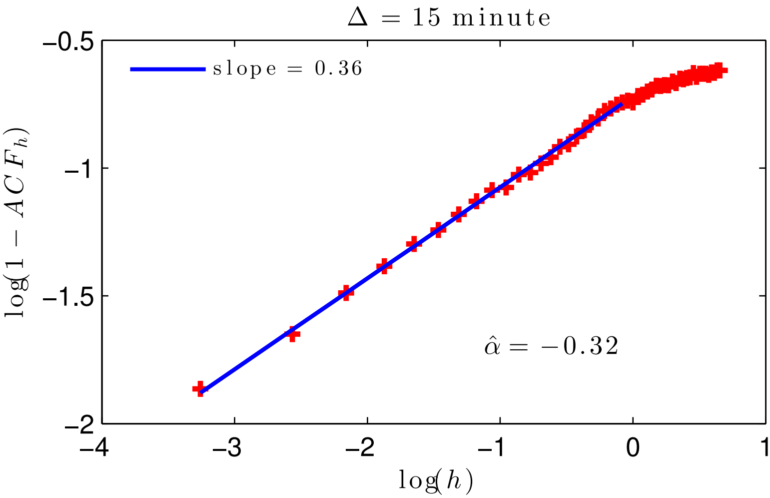
1. Inference (hypothesis testing, confidence intervals): Fractional bootstrap method of MB (2016-WP).
2. Swept under the rug: $\widehat{\sigma_t^2}$ is a noisy estimator of σ_t^2...

Estimating roughness: E-mini S&P 500 data



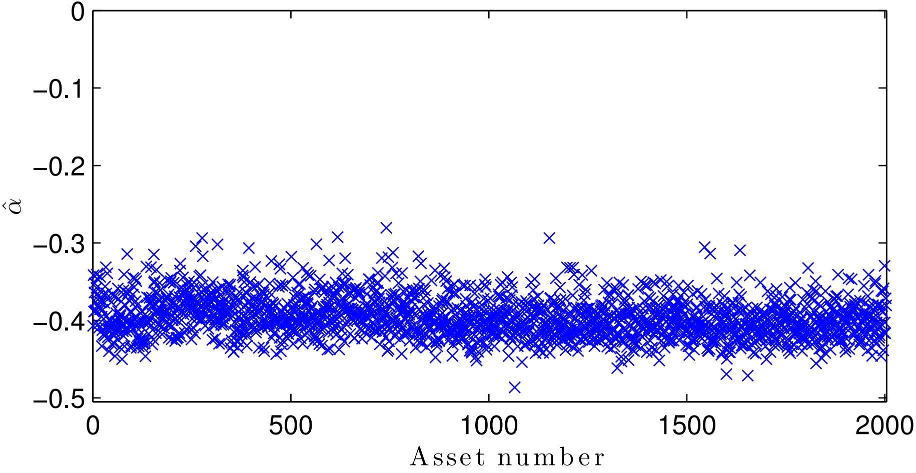
TAQ data

We study high-frequency data from the Trades-and-Quotes (TAQ) database.

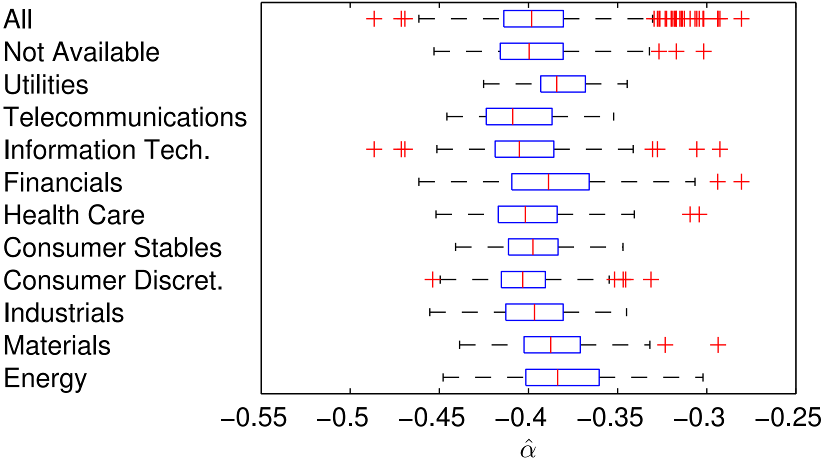
- Total number of assets: $10,064$.
- Divided into 10 sectors.
- Take only most liquid ones:
  - Period: Q1 2003 - Q4 2013.
  - Minimum 400 days in sample.
  - Traded every 5 minutes (on average).
  - Assets left: $2,003$

We take these assets, calculate daily RK and estimate α.

Universal roughness: liquid assets



Roughness by sector: liquid assets



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Remarks

1. If $\beta < 1$ then X has long memory.
2. The H-fBm has $H = 1 - \beta/2 = \alpha + 1/2$.

Estimating persistence

Parametic estimator of β

Assume a parametric form of the ACF, e.g. the Cauchy class

$$\rho(h) = \left(1 + |h|^{2\alpha+1}\right)^{-\beta/(2\alpha+1)}$$

Then, plug in $\hat{\alpha}_{OLS}$ and fit the theoretical ACF to the empirical:

$$\hat{\beta}_{Cauchy} := \arg\min_{\beta>0} \frac{1}{m} \sum_{k=1}^{m} |\rho(k\Delta; \beta) - \hat{\rho}(k\Delta)|^2.$$

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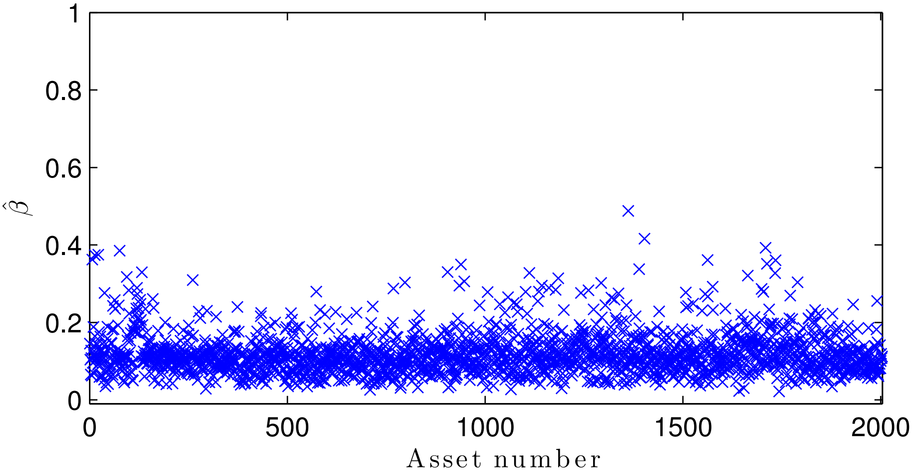
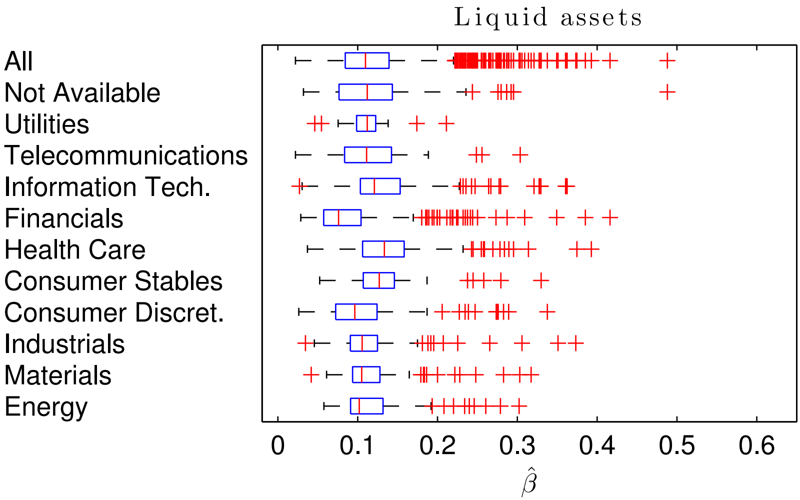
2. Parametric and semiparametric largely agree.

Universal persistence: liquid assets



Persistence by sector: liquid assets



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- We suppose

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- We want X to be:
  - Stationary.
  - Rough.
  - Persistent.
  - Possibly non-Gaussian.

The Brownian semistationary ($\mathcal{BSS}$) process

Definition: $\mathcal{BSS}$ process

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Main examples:

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Main examples:

1. $g(x) = x^{\alpha} e^{-\lambda x}, \quad \lambda > 0$ (Gamma-kernel; Matérn correlation)
2. $g(x) = x^{\alpha}(1+x)^{-\gamma-\alpha}, \quad \gamma > 1/2$ (Power-kernel; persistent)

Some properties of $\mathcal{BSS}$ processes I

- Define $\rho_X(h) := Corr(X_t, X_{t+h})$.

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Corollary: Hölder continuity

The $\mathcal{BSS}$ process has a modification with locally ϕ-Hölder continuous trajectories for any $\phi \in (0, \alpha + 1/2)$.

Some properties of $\mathcal{BSS}$ processes II

Proposition: The $\mathcal{BSS}$ process has short memory

- Let

$$g(x) = e^{-\lambda x} x^{-\gamma} L_1(x), \quad x \in (1, \infty), \quad \lambda > 0, \gamma \in \mathbb{R}.$$

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Some properties of $\mathcal{BSS}$ processes III

Proposition: The $\mathcal{BSS}$ process is persistent

- Let

$$g(x) = x^{-\gamma} L_1(x), \quad x \in (1, \infty), \quad \gamma > 1/2.$$

1. If $\gamma > 1$, then
$$\rho_X(h) \sim h^{-\gamma} L_1(h), \quad h \to \infty.$$
2. If $\gamma \in (1/2, 1)$, then
$$\rho_X(h) \sim h^{1-2\gamma} L_1(h)^2, \quad h \to \infty.$$

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Remark

1. For $\gamma \in (1/2, 1)$ we have $2\gamma - 1 \in (0, 1)$ so that the process has long memory.

Properties of raw volatility

Theorem: implications for raw volatility

Suppose X is a Gaussian $\mathcal{BSS}$ process. Set $\sigma_t = \xi \exp(X_t)$ and let

$$\rho(h) := Corr(\sigma_{t+h}, \sigma_t), \quad t, h \in \mathbb{R}.$$

1. As $h \to 0$,

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2. As $h \to \infty$,

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Remarks

1. Both roughness and persistent is found in raw volatility also.
2. We conjecture that this holds for more general $\mathcal{BSS}$ processes as well, under suitable assumptions.

The $\mathcal{BSS}$ process as a model of log-vol

The $\mathcal{BSS}$ process...

1. ...is stationary.
2. ...is non-Gaussian.
   - $X_t|v \sim N\left(0, \int_0^\infty g(x)^2 v_{t-x}^2 dx\right)$.
3. ...decouples short- and long-term behavior:
   - Roughness.
   - Persistent/long memory.
4. ...is easy to...
   - estimate (OLS, method of moments).
   - simulate (see talk by Mikko S. Pakkanen Wednesday).
   - forecast (Lunde, Pakkanen and MB., 2016-WP).
   - include leverage.

Outline

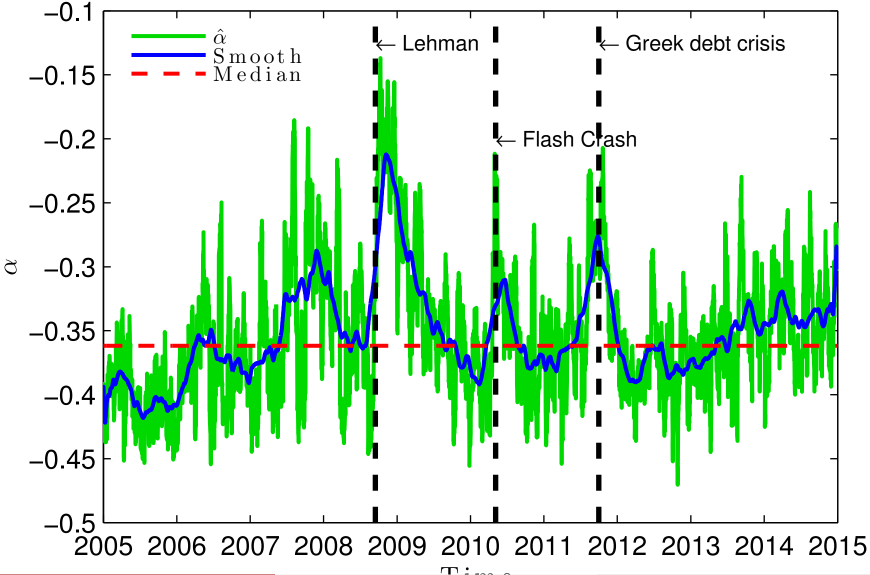
1. Empirical evidence
   - Roughness of log-volatility
   - Persistence of log-volatility
2. Modelling log-volatility by $\mathcal{BSS}$ processes
3. Final remarks

Does roughness change over time?

One can ask, whether the roughness of volatility is constant over time.

- We study the E-mini S&P 500 data from 2005 to 2015:

Does roughness change over time?



An open problem: prediction

- For the H-fBm B^H with $H > 1/2$ it holds (Nuzman and Poor, 2000-JoAP)

$$\mathbb{E}[B_{t+h}^H | \mathcal{F}_t] = \frac{\cos(H\pi)}{\pi} h^{H+1/2} \int_{-\infty}^{t} \frac{B_s^H}{(t-s+h)(t-s)^{H+1/2}} ds.$$

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- While for the $\mathcal{BSS}$ process $X_t = \int_{-\infty}^t g(t-s) dW_s$,

$$\mathbb{E}[X_{t+h}|\mathcal{F}_t] = (\text{Open problem}).$$

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2. The $\mathcal{BSS}$ process is an ideal candidate as a model of log-vol.
    - Very tractable (ACF, estimation, simulation, ...).
    - Very good empirical fit (roughness, persistence, non-Gaussian, ...).
    - Very good at forecasting.

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Remark

1. Actually, its forecasting prowess is one of the main reasons for us advocating the $\mathcal{BSS}$ process as a model of volatility!

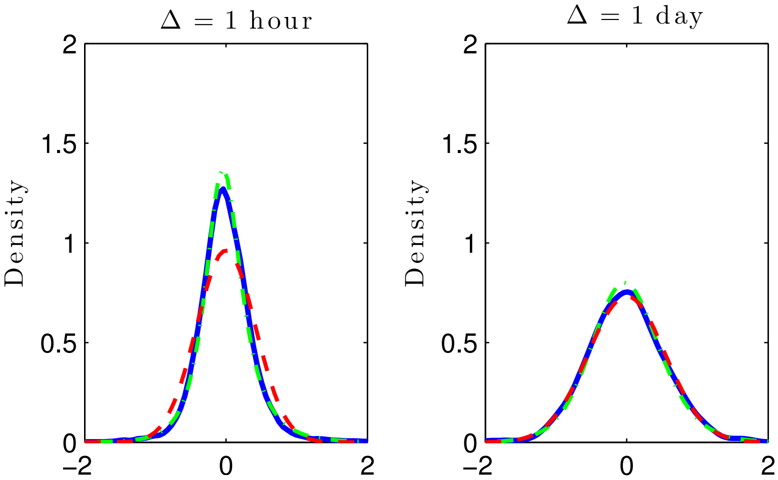
Thank you

Thank you for your attention!

Thank you

APPENDIX

Non-Gaussianity of log-volatility



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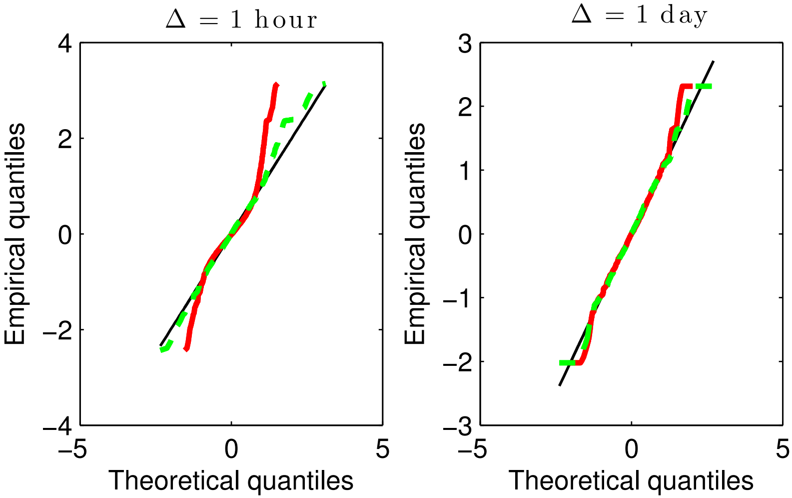
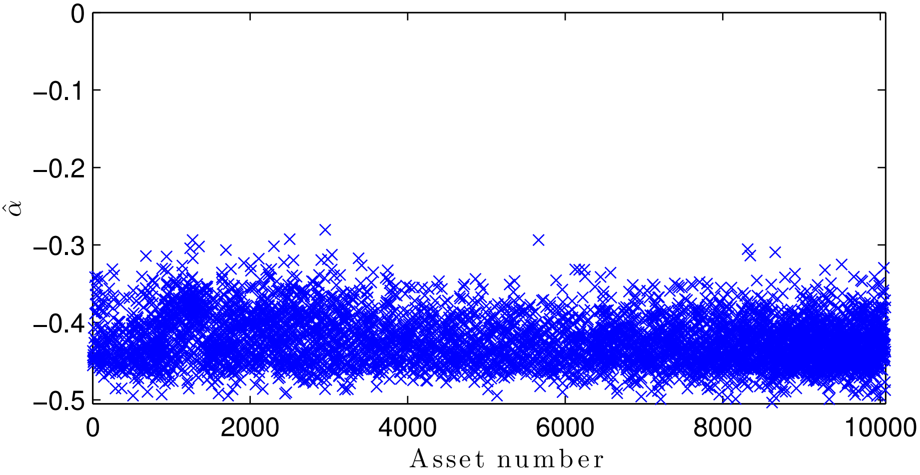
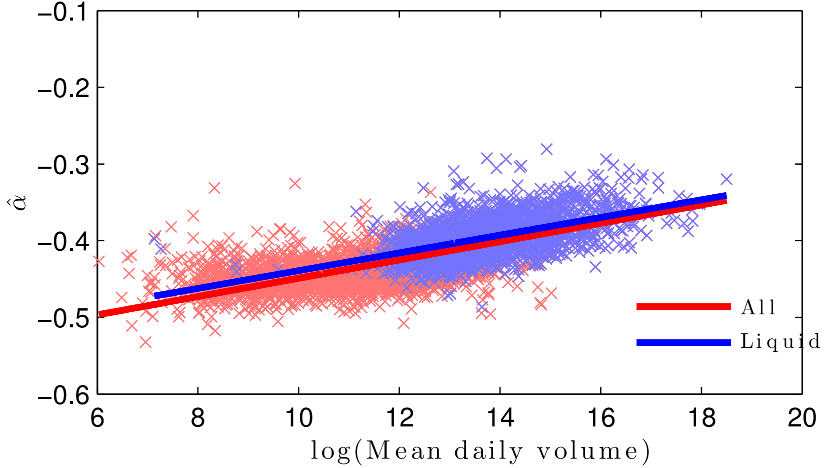


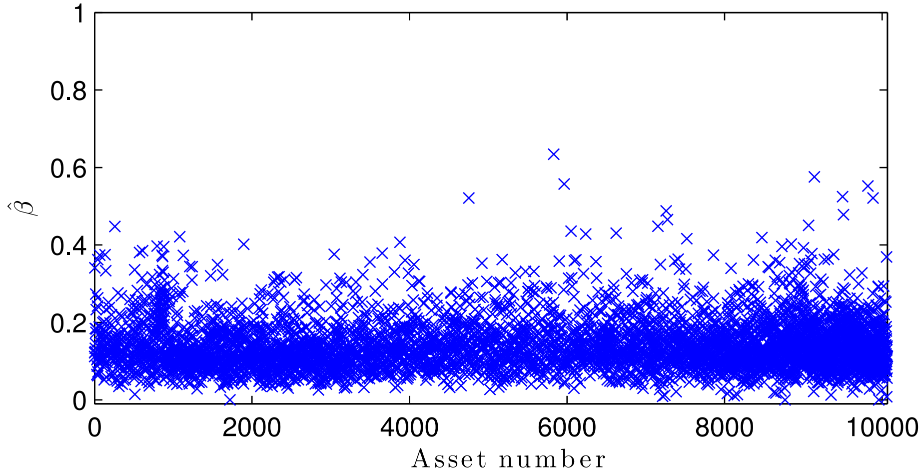
Universal roughness: all assets



Universal roughness: roughness vs. volume



Universal persistence: all assets



Forecasting

FORECASTING

Forecasting intraday volatility in E-mini data

Forecasting setup:

- Sample period: January 2, 2013 until December 31, 2014.
- Sample frequencies: $\Delta = 2, 5, 10, 15, 30$ minutes and $\Delta = 1, 2, 6.5$ hours.
- Forecast horizons: $h = 1, 2, 5, 10, 20$ periods.
- Benchmark models: RW, AR(1), AR(5), AR(10), ARMA, HAR, ARFIMA, RFSV.
- Loss functions: MAE, MSE, QLIKE.
- Also: Model Confidence Set (Hansen et al. 2011-E).

Forecasting results:

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Forecasting results:

1. Our model(s) win in almost all cases (in- and out-sample).

Out-of-sample forecasting of E-mini: $\Delta = 60$ minutes

	$h = 1$		
	MAE $\times 10^5$	MSE $\times 10^{10}$	QLIKE
RW	0.193	0.181	-11.248^*
AR1	0.196	0.144	-11.242^*
AR5	0.193	0.150	-11.246^*
AR10	0.195	0.153	-11.246^*
HAR3	0.216	0.168	$\mathbf{-11.226}$
ARFIMA	0.183	0.134	-11.250
RFSV	0.183	0.144	-11.260
BSS	$\mathbf{0.175}^*$	$\mathbf{0.127}^*$	-11.260

Out-of-sample forecasting of E-mini: $\Delta = 60$ minutes

	$h = 5$		
	MAE $\times 10^4$	MSE $\times 10^9$	QLIKE
RW	0.114	0.580	-9.525
AR1	0.147	0.480	-9.192
AR5	0.103	0.282	-9.548
AR10	0.103	0.293	-9.550
HAR3	0.106	0.294^*	-9.545
ARFIMA	0.097	0.241	-9.553
RFSV	0.101	0.289	-9.559
BSS	**0.091^***	**0.215^***	**-9.565^***

Estimation

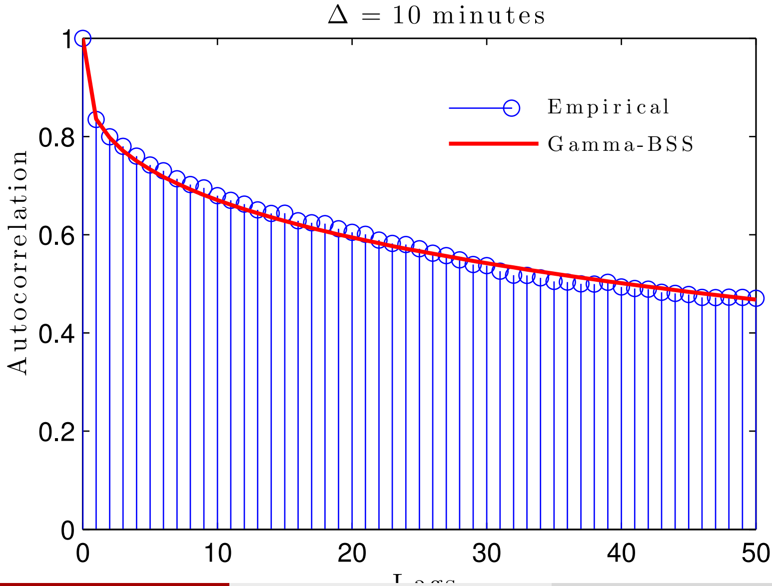
ESTIMATION

Estimation of the models

Data: Log-volatility

Δ	$\hat{\alpha}_{OLS}$	95% CI	$\hat{\lambda}_{BSS}$	$\hat{\beta}_{OLS}$	$\hat{\beta}_{BSS}$	$\hat{\beta}_{Cauchy}$
1 min	-0.48	$(-0.49, -0.46)$	0.01	0.08	0.12	0.09
2 mins	-0.46	$(-0.47, -0.46)$	0.00	0.11	0.17	0.09
5 mins	-0.41	$(-0.42, -0.39)$	0.03	0.15	0.28	0.15
10 mins	-0.35	$(-0.37, -0.34)$	0.10	0.19	0.45	0.23
15 mins	-0.32	$(-0.34, -0.31)$	0.13	0.27	0.57	0.30
30 mins	-0.30	$(-0.37, -0.26)$	0.12	0.24	0.60	0.34
1 hour	-0.31	$(-0.45, -0.20)$	0.10	0.19	0.48	0.33
2 hours	-0.32	$(-0.45, -0.23)$	0.08	0.31	0.46	0.34
1 day	-0.28	$(-0.36, -0.24)$	0.12	1.08	0.59	0.60

The autocorrelation of log-volatility: $\Delta = 10$ minutes



The autocorrelation of log-volatility: $\Delta = 1$ day