A Malliavin-Skorohod calculus in $L^0$ and $L^1$ for pure jump additive and Volterra-type processes

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Abstract

- In this paper we extend the Malliavin-Skorohod type calculus for pure jump additive processes to the $L^0$ and $L^1$ settings.
- We apply it to extend stochastic integration with respect to volatility modulated pure jump additive-driven Volterra processes.
- In particular, we define integrals with respect to Volterra processes driven by $\alpha$-stable processes with $\alpha < 2$. 
Motivation I

Consider a pure jump volatility modulated additive driven Volterra (VMAV) process $X$ defined as

$$X(t) = \int_0^t g(t, s)\sigma(s)dJ(s)$$

provided the integral is well defined. Here $J$ is a pure jump additive process, $g$ is a deterministic function and $\sigma$ is a predictable process with respect the natural completed filtration of $J$.

This kind of models, called volatility modulated Volterra processes, are part of the family of Ambit processes and are used in modeling turbulence, energy finance and others.
A major problem is to develop an integration theory with respect $X$ as integrator, that is, to give a meaning to

$$\int_0^t Y(s)dX(s)$$

for a fixed $t$ and a suitable stochastic processes $Y$. Recall that $X$ is not necessarily a semimartingale.

This has been done in [BBPV], assuming $J$ is a square integrable pure jump Lévy process and assuming Malliavin regularity conditions on $Y$ in the $L^2$ setting.
Here we extend this integration theory to any pure jump additive process, not necessary square integrable, and in particular allowing to treat integration, for example, with respect to $\alpha$-stable processes when $\alpha < 2$.

Integrability conditions related with $Y$ are in the $L^1$ setting. So, our results are an extension on the previous ones in the finite activity case and treat new cases in the infinite activity case.
The Malliavin-Skorohod calculus for square integrable functionals of an additive process is today a well established topic. See for example Yablonski (2008).

In [SUV] a new canonical space for Lévy processes is introduced and a probabilistic interpretation of Malliavin-Skorohod operators in this space is obtained.

These operators defined in the canonical space are well defined beyond the $L^2$ setting.
This allows to explore the development of a Malliavin-Skorohod calculus for functionals adapted to a general additive processes that belong only to $L^1$ or $L^0$.

This is the main goal of our work, that can be seen as an extension of [SUV] using also ideas from Picard (1996).

In particular we prove several rules of calculus and a new version of the Clark-Haussmann-Ocone (CHO) formula in the $L^1$ setting.
Let $X = \{X_t, t \geq 0\}$ be an additive process, that is, a process with independent increments, stochastically continuous, null at the origin and with càdlàg trajectories.

Let $\mathbb{R}_0 := \mathbb{R} - \{0\}$.

For any fixed $\epsilon > 0$, denote $S_\epsilon := \{|x| > \epsilon\} \subseteq \mathbb{R}_0$.

Let us denote $\mathcal{B}$ and $\mathcal{B}_0$ the $\sigma$–algebras of Borel sets of $\mathbb{R}$ and $\mathbb{R}_0$ respectively.
The distribution of an additive process can be characterized by the triplet \((\Gamma_t, \sigma_t^2, \nu_t), \ t \geq 0\), where

- \(\{\Gamma_t, t \geq 0\}\) is a continuous function null at the origin.
- \(\{\sigma_t^2, t \geq 0\}\) is a continuous and non-decreasing function null at the origin.
- \(\{\nu_t, t \geq 0\}\) is a set of Lévy measures on \(\mathbb{R}\). Moreover, for any set \(B \in \mathcal{B}_0\) such that \(B \subseteq S_\epsilon\) for a certain \(\epsilon > 0\), \(\nu.(B)\) is a continuous and increasing function null at the origin.
Let $\Theta := [0, \infty) \times \mathbb{R}$. Denote $\theta := (t, x) \in \Theta$ and $d\theta = (dt, dx)$.

For $T \geq 0$, we introduce the measurable spaces $(\Theta_{T,\epsilon}, \mathcal{B}(\Theta_{T,\epsilon}))$ where $\Theta_{T,\epsilon} := [0, T] \times S_{\epsilon}$.

Observe that $\Theta_{\infty,0} = [0, \infty) \times \mathbb{R}_0$ and that $\Theta$ can be represented as $\Theta = \Theta_{\infty,0} \cup ([0, \infty) \times \{0\})$. 
We introduce a measure \( \nu \) on \( \Theta_{\infty,0} \) such that for any \( B \in \mathcal{B}_0 \) we have \( \nu([0, t] \times B) := \nu_t(B) \). The hypotheses on \( \nu_t \) guarantee that \( \nu(\{t\} \times B) = 0 \) for any \( t \geq 0 \) and for any \( B \in \mathcal{B}_0 \). Note that in particular, \( \nu \) is \( \sigma \)-finite.

Let \( N \) be the jump measure associated to \( X \). Recall that it is a Poisson random measure on \( \mathcal{B}(\Theta_{\infty,0}) \) with parameter \( \nu \). Denote \( \tilde{N}(dt, dx) := N(dt, dx) - \nu(dt, dx) \).

We can introduce also a \( \sigma \)-finite measure \( \sigma \) on \([0, \infty)\) such that \( \sigma([0, t]) = \sigma_t^2 \).
According to the Lévy-Itô decomposition we can write:

\[ X_t = \Gamma_t + W_t + J_t, \quad t \geq 0 \]

where

- \( \Gamma \) is a continuous deterministic function null at the origin.
- \( W \) is a centered Gaussian process with variance process \( \sigma^2 \).
J is an additive process with triplet \((0, 0, \nu_t)\) independent of \(W\), defined by

\[
J_t = \int_{\Theta_t,1} xN(ds, dx) + \lim_{\epsilon \downarrow 0} \int_{\Theta_t,\epsilon - \Theta_t,1} x\tilde{N}(ds, dx)
\]

where the convergence is a.s. and uniform with respect to \(t\) on every bounded interval. We call the process \(J = \{J_t, t \geq 0\}\) a pure jump additive process.

Moreover, if \(\{\mathcal{F}_t^W, t \geq 0\}\) and \(\{\mathcal{F}_t^J, t \geq 0\}\) are, respectively, the completed natural filtrations of \(W\) and \(J\), then, for every \(t \geq 0\), we have \(\mathcal{F}_t^X = \mathcal{F}_t^W \vee \mathcal{F}_t^J\).
We consider on $\Theta$ the $\sigma-$finite Borel measure

$$
\mu(dt, dx) := \sigma(dt)\delta_0(dx) + \nu(dt, dx).
$$

Note that $\mu$ is continuous in the sense that $\mu(\{t\} \times B) = 0$ for all $t \geq 0$ and $B \in \mathcal{B}$.

Then we define

$$
M(dt, dx) = (W \otimes \delta_0)(dt, dx) + \tilde{N}(dt, dx)
$$

that is a centered random measure with independent values such that

$$
\mathbb{E}[M(E_1)M(E_2)] = \mu(E_1 \cap E_2), \text{ for } E_1, E_2 \in \mathcal{B}(\Theta) \text{ with } \mu(E_1) < \infty \text{ and } \mu(E_2) < \infty.
$$
If we take $\sigma^2 \equiv 0$, $\mu = \nu$ and $M = \tilde{N}$, we recover the Poisson random measure case.

If we take $\nu = 0$, we have $\mu(dt, dx) := \sigma(dt)\delta_0(dx)$ and $M(ds, dx) = (W \otimes \delta_0)(ds, dx)$ and we recover the independent increment centered Gaussian measure case.

If we take $\sigma_t^2 := \sigma_L^2 t$ and $\nu(dt, dx) = dt\nu_L(dx)$, we obtain $M(ds, dx) = \sigma_L(W \otimes \delta_0)(ds, dx) + \tilde{N}(ds, dx)$ and we recover the Lévy case (stationary increments case).
Malliavin-Skorohod calculus for additive processes in $L^2$.

We recall the presentation of the Malliavin-Skorohod calculus with respect to the random measure $M$ on its canonical space in the $L^2$–framework, as a first step towards our final goal of extending the calculus to the $L^1$ and $L^0$ frameworks.
The chaos representation property

- Given $\mu$, we can consider the spaces

\[
\mathbb{L}_n^2 := L^2\left(\Theta^n, \mathcal{B}(\Theta)^{\otimes n}, \mu^{\otimes n}\right)
\]

and define for functions $f$ in $\mathbb{L}_n^2$ the Itô multiple stochastic integrals $I_n(f)$ with respect to $M$ in the usual way.

- Then we have the so-called chaos representation property, that is, for any functional $F \in L^2(\Omega, \mathcal{F}^X, \mathbb{P})$, where $\mathcal{F}^X = \bigvee_{t \geq 0} \mathcal{F}_t^X$, we have

\[
F = \sum_{n=0}^{\infty} I_n(f_n)
\]

for a certain unique family of symmetric kernels $f_n \in \mathbb{L}_n^2$. 

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The Malliavin and Skorohod operators I

The chaos representation property of $L^2(\Omega, \mathcal{F}^X, \mathbb{P})$ shows that this space has a Fock space structure. Thus it is possible to apply all the machinery related to the anhilation operator (Malliavin derivative) and the creation operator (Skorohod integral).

Consider $F = \sum_{n=0}^{\infty} l_n(f_n)$, with $f_n$ symmetric and such that $\sum_{n=1}^{\infty} n^n ||f_n||_{L^2_n}^2 < \infty$. The Malliavin derivative of $F$ is an object of $L^2(\Theta \times \Omega, \mu \otimes \mathbb{P})$, defined as

$$D_\theta F = \sum_{n=1}^{\infty} n l_{n-1} \left( f_n(\theta, \cdot) \right), \; \theta \in \Theta.$$ 

We denote by $\text{Dom}D$ the domain of this operator.
Let \( u \in L^2(\Theta \times \Omega, \mathcal{B}(\Theta) \otimes \mathcal{F}^X, \mu \otimes \mathbb{P}) \). For every \( \theta \in \Theta \) we have the chaos decomposition

\[
    u_\theta = \sum_{n=0}^{\infty} l_n(f_n(\theta, \cdot))
\]

where \( f_n \in \mathbb{L}^2_{n+1} \) is symmetric in the last \( n \) variables. Let \( \tilde{f}_n \) be the symmetrization in all \( n + 1 \) variables. Then we define the Skorohod integral of \( u \) by

\[
    \delta(u) = \sum_{n=0}^{\infty} l_{n+1}(\tilde{f}_n),
\]

in \( L^2(\Omega) \), provided \( u \in \text{Dom} \delta \), that means

\[
    \sum_{n=0}^{\infty} (n + 1)! \| \tilde{f}_n \|_{\mathbb{L}^2_{n+1}}^2 < \infty.
\]
Duality between the Malliavin and Skorohod operators

- If \( u \in \text{Dom}\ \delta \) and \( F \in \text{Dom}\ D \) we have the duality relation

\[
\mathbb{E}[\delta(u) F] = \mathbb{E} \int_{\Theta} u_\theta D_\theta F \mu(d\theta).
\]

- We recall that if \( u \in \text{Dom}\ \delta \) is actually predictable with respect to the filtration generated by \( X \), then the Skorohod integral coincides with the (non anticipating) Itô integral in the \( L^2 \)—setting with respect to \( M \).
Let $A \in \mathcal{B}(\Theta)$ and $\mathcal{F}_A := \sigma\{M(A') : A' \in \mathcal{B}(\Theta), A' \subseteq A\}$.

- $F$ is $\mathcal{F}_A$–measurable if for any $n \geq 1$, $f_n(\theta_1, \ldots, \theta_n) = 0$, $\mu^\otimes n$ – a.e. unless $\theta_i \in A \ \forall \ i = 1, \ldots, n$.

- In particular, we are interested in the case $A = \Theta_t := [0, t) \times \mathbb{R}$. Denote $\mathcal{F}_{t-} := \mathcal{F}_{\Theta_{t-}}$. Obviously, if $F \in \text{Dom } D$ and it is $\mathcal{F}_{t-}$–measurable then $D_{s,x} F = 0$ for a.e. $s \geq t$ and any $x \in \mathbb{R}$. 
The Clark-Haussmann-Ocone formula II

From the chaos representation property we can see that for $F \in L^2(\Omega)$,

$$E[F|\mathcal{F}_{t-}] = \sum_{n=0}^{\infty} l_n(f_n(\theta_1, \ldots, \theta_n) \prod_{i=1}^{n} 1_{[0,t]}(t_i)).$$

Then, for $F \in \text{Dom}D$ we have

$$D_{s,x}E[F|\mathcal{F}_{t-}] = E[D_{s,x}F|\mathcal{F}_{t-}] 1_{[0,t]}(s), (s, x) \in \Theta.$$
The Clark-Haussmann-Ocone formula III

Using these facts we can prove the very well known CHO formula: If $F \in \text{Dom}D$ we have

$$F = \mathbb{E}(F) + \delta(E[D_{t,x}F | \mathcal{F}_{t-}]).$$

- Note that being the integrand a predictable process, the Skorohod integral $\delta$ here above is actually an Itô integral.
- Note also that the CHO formula can be rewritten in a decompactified form as

$$F = \mathbb{E}(F) + \int_0^\infty E(D_{s,0}F | \mathcal{F}_{s-})dW_s + \int_{\Theta_{\infty,0}} E(D_{s,x}F | \mathcal{F}_{s-})\tilde{N}(ds, dx).$$
We set our work on the canonical space of $J$, substantially introduced in [SUV].

The construction is done first of all in the case $\nu$ is concentrated on $\Theta_{T,\epsilon}$ for a fixed $T > 0$ and $\epsilon > 0$, that is a finite activity case. Later the construction is extended to the case $\Theta_{\infty,0}$ taking $T \uparrow \infty$ and $\epsilon \downarrow 0$.

In the case $\nu$ concentrated on $\Theta_{T,\epsilon}$, and so finite, any trajectory of $J$ can be totally described by a finite sequence

$$(t_1, x_1), \ldots, (t_n, x_n)$$

where $t_1, \ldots, t_n \in [0, T]$ are the jump instants, with $t_1 < t_2 < \cdots < t_n$, and $x_1, \ldots, x_n \in S_{\epsilon}$ are the corresponding sizes, for some $n$. 

A canonical space for $J$ II

- The extension to the space $\Theta_{\infty,0}$ is done through a projective system of probability spaces.
- For every $m \geq 1$ we consider the probability spaces

\[
(\Omega^J_m, \mathcal{F}_m, \mathbb{P}_m) := (\Omega^J_{\frac{1}{m}}, \mathcal{F}_{\frac{1}{m}}, \mathbb{P}_{\frac{1}{m}}),
\]

that are the canonical spaces corresponding to $\Theta_m := [0, m] \times S_{\frac{1}{m}}$.
- Then the canonical space $\Omega^J$ for $J$ on $\Theta_{\infty,0}$ is defined as the projective limit of the system $(\Omega_m^J, m \geq 1)$.
In our setup, $\Omega^J = \bigcup_{n=0}^{\infty} \Theta^n_{\infty,0}$ and the probability measure $\mathbb{P}$ is concentrated on the subset of

- The empty sequence $\alpha$, corresponding to the element $(\alpha, \alpha, \ldots)$.
- All finite sequences of pairs $(t_i, x_i)$.
- All infinite sequences of pairs $(t_i, x_i)$ such that for every $m > 0$ there is only a finite number of $(t_i, x_i)$ on $\Theta_m$. 

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Now we establish the basis for a Malliavin-Skorohod calculus with respect to a pure jump additive process, constructively on the canonical space.

In general, the proofs of the following results are done directly on $\Omega_m^J$ and extended to $\Omega^J$ by dominated convergence.
Let $\theta = (s, x) \in \Theta_{\infty, 0}$. Let $\omega \in \Omega^J$, that is, $\omega := (\theta_1, \ldots, \theta_n, \ldots)$, with $\theta_i := (s_i, x_i)$.

We introduce the following two transformations from $\Theta_{\infty, 0} \times \Omega^J$ to $\Omega^J$:

$$\epsilon_\theta^+ \omega := ((s, x), (s_1, x_1), (s_2, x_2), \ldots),$$

where a jump of size $x$ is added at time $s$, and

$$\epsilon_\theta^- \omega := ((s_1, x_1), (s_2, x_2), \ldots) - \{(s, x)\},$$

where we take away the point $\theta = (s, x)$ from $\omega$. 
Properties of the Transformations

These two transformations are analogous to the ones introduced in Picard (1996).

Observe that $\epsilon^+\omega$ is well defined except on the set $\{(\theta, \omega) : \theta \in \omega\}$ that has null measure with respect $\nu \otimes P$. We can consider by convention that on this set, $\epsilon^+\omega := \omega$.

The case of $\epsilon_-\omega$ is also clear. In fact this operator satisfies $\epsilon^-\omega = \omega$ except on the set $\{(\theta, \omega) : \theta \in \omega\}$.

For simplicity of the notation sometimes we will denote $\hat{\omega}_i := \epsilon^-\omega$. 
For a random variable $F \in L^0(\Omega^J)$, we define the operator

$$T : L^0(\Omega^J) \mapsto L^0(\Theta_\infty, 0 \times \Omega^J),$$

such that $(T_\theta F)(\omega) := F(\epsilon_\theta^+ \omega)$.

It is not difficult to see that if $F$ is a $\mathcal{F}^J$-measurable, then

$$(T.F)(\cdot) : \Theta_\infty, 0 \times \Omega^J \longrightarrow \mathbb{R}$$

is $\mathcal{B}(\Theta_\infty, 0) \otimes \mathcal{F}^J$-measurable and $F = 0$, $\mathbb{P}$-a.s. implies $T.F(\cdot) = 0$, $\nu \otimes \mathbb{P}$-a.e. So, $T$ is a closed linear operator defined on the entire $L^0(\Omega^J)$. 


The operator $T$ II

But if we want to assure $T.F(\cdot) \in L^1(\Theta_\infty,0 \times \Omega^J)$ we have to restrict the domain and guarantee that

$$
\mathbb{E} \int_{\Theta_\infty,0} |T_\theta F| \nu(d\theta) < \infty.
$$

This requires a condition that is strictly stronger than $F \in L^1(\Omega^J)$. 
The operator $T_{III}$

Concretely, denoting $k_m := e^{-\nu(\Theta_m - \Theta_{m-1})}$, we have to assume that

$$
\sum_{m=1}^{\infty} k_m \sum_{n=0}^{\infty} \frac{n}{n!} \int_{(\Theta_m - \Theta_{m-1})^n} |F(\theta_1, \ldots, \theta_n)| \nu(d\theta_1) \ldots \nu(d\theta_n) < \infty,
$$

whereas $F \in L^1(\Omega)$ is equivalent only to

$$
\sum_{m=1}^{\infty} k_m \sum_{n=0}^{\infty} \frac{1}{n!} \int_{(\Theta_m - \Theta_{m-1})^n} |F(\theta_1, \ldots, \theta_n)| \nu(d\theta_1) \ldots \nu(d\theta_n) < \infty.
$$
For a random field $u \in L^0(\Theta_\infty, 0 \times \Omega^J)$ we define the operator

$$S : \text{Dom} S \subseteq L^0(\Theta_\infty, 0 \times \Omega^J) \rightarrow L^0(\Omega^J)$$

such that

$$(Su)(\omega) := \int_{\Theta_\infty,0} u_\theta(\epsilon^- \omega) N(d\theta, \omega) := \sum_i u_{\theta_i}(\hat{\omega}_i) < \infty.$$ 

In particular, if $\omega = \alpha$, we define $(Su)(\alpha) = 0.$
The operator $S$ is well defined and closed from $L^1(\Theta_{\infty,0} \times \Omega^J)$ to $L^1(\Omega)$ as the following proposition says:

**Proposition**

If $u \in L^1(\Theta_{\infty,0} \times \Omega^J)$, $Su$ is well defined and takes values in $L^1(\Omega)$. Moreover

$$
\mathbb{E} \int_{\Theta_{\infty,0}} u_{\theta}(\varepsilon_{\theta}^{-} \omega) N(d\theta, \omega) = \mathbb{E} \int_{\Theta_{\infty,0}} u_{\theta}(\omega) \nu(d\theta).
$$
THE OPERATOR $S$ III

Given $\theta = (s, x)$ we can define for any $\omega, \tilde{\omega}_s$ as the $\omega$ restricted to jump instants strictly before $s$. In this case, obviously, $\epsilon_\theta \tilde{\omega}_s = \tilde{\omega}_s$. If $u$ is predictable we have $u_\theta(\omega) = u_\theta(\tilde{\omega}_s)$ and so

$$u_\theta(\epsilon_\theta \omega) = u_\theta(\omega),$$

and

$$(Su)(\omega) = \int_{\Theta_{\infty, 0}} u_\theta(\epsilon_\theta \omega) N(d\theta, \omega) = \int_{\Theta_{\infty, 0}} u_\theta(\omega) N(d\theta, \omega).$$
The following theorem is the fundamental relationship between operators $S$ and $T$:

**Theorem**

Consider $F \in L^0(\Omega^J)$ and $u \in \text{Dom} S$. Then $F \cdot Su \in L^1(\Omega^J)$ if and only if $TF \cdot u \in L^1(\Theta_{\infty,0} \times \Omega^J)$ and in this case

$$
\mathbb{E}(F \cdot Su) = \mathbb{E} \int_{\Theta_{\infty,0}} T_\theta F \cdot u_\theta \nu(d\theta).
$$
Rules of calculus

- If $u$ and $TF \cdot u$ belong to $DomS$ we have
  
  $$F \cdot Su = S(TF \cdot u), \ P - a.e.$$  

- If $u$ and $Tu$ are in $DomS$ then
  
  $$T_\theta(Su) = u_\theta + S(T_\theta u), \ \nu \otimes P - a.e.$$
Now we introduce the operator $\psi_{t,x} := T_{t,x} - Id$. Observe that this operator is linear, closed and satisfies the property

$$\psi_{t,x}(F G) = G \psi_{t,x} F + F \psi_{t,x} G + \psi_{t,x}(F) \psi_{t,x}(G).$$
The operator $\mathcal{E}$

On other hand, for $u \in L^0(\Theta_{\infty}, 0 \times \Omega^J)$ we consider the operator:

$$\mathcal{E} : \text{Dom}\mathcal{E} \subseteq L^0(\Theta_{\infty}, 0 \times \Omega^J) \longrightarrow L^0(\Omega^J)$$

such that

$$(\mathcal{E} u)(\omega) := \int_{\Theta_{\infty}, 0} u_\theta(\omega) \nu(d\theta).$$

Note that $\text{Dom}\mathcal{E}$ is the subset of processes in $L^0(\Theta_{\infty}, 0 \times \Omega^J)$ such that $u(\cdot, \omega) \in L^1(\Theta_{\infty}, 0), \mathbb{P} - \text{a.e.}$

We have also that

$$\int_{\Theta_{\infty}, 0} u_\theta(\epsilon_\theta^\omega) \nu(d\theta) = \int_{\Theta_{\infty}, 0} u_\theta(\omega) \nu(d\theta), \mathbb{P} - \text{a.s.}$$
The operator $\Phi$

Then, for $u \in Dom\Phi := DomS \cap Dom\mathcal{E} \subseteq L^0(\Theta_\infty,0 \times \Omega^J)$, we define

$$\Phi u := Su - \mathcal{E}u.$$ 

Note that

- $L^1(\Theta_\infty,0 \times \Omega^J) \subseteq Dom\Phi$.
- $E(\Phi u) = 0$, for any $u \in L^1(\Theta_\infty,0 \times \Omega)$.
- For any $u \in Dom\Phi$, predictable,

$$\Phi(u) = \int_{\Theta_\infty,0} u_\theta(\omega) \tilde{N}(d\theta,\omega).$$

- $u \in L^2(\Theta_\infty,0 \times \Omega^J)$ not implies $u \in L^1(\Theta_\infty,0 \times \Omega^J)$ nor $u \in Dom\Phi$. 

As a corollary of the duality between $T$ and $S$ we have the following result:

**Proposition**

Consider $F \in L^0(\Omega^J)$ and $u \in \text{Dom}\Phi$. Assume also $F \cdot u \in L^1(\Theta_\infty,0 \times \Omega^J)$. Then $F \cdot \Phi u \in L^1(\Omega^J)$ if and only if $\Psi F \cdot u \in L^1(\Theta_\infty,0 \times \Omega^J)$ and in this case

$$
\mathbb{E}(F \cdot \Phi u) = \mathbb{E}\left( \int_{\Theta_\infty,0} \Psi \theta F \cdot u_\theta \nu(d\theta) \right).
$$
Rules of calculus

- If $F \in L^0(\Omega^J)$ and $u$, $F \cdot u$ and $\Psi F \cdot u$ belong to $\text{Dom}\Phi$ we have
  \[ \Phi(F \cdot u) = F \cdot \Phi u - \Phi(\Psi F \cdot u) - \mathcal{E}(\Psi F \cdot u), \ P - \text{a.s.} \]

- If $u$ and $\Psi u$ belong to $\text{Dom}\Phi$ we have
  \[ \Psi_\theta(\Phi u) = u_\theta + \Phi(\Psi_\theta u), \ \nu \otimes P - \text{a.e.} \]
Consider now the operators $D$ and $\delta$ restricted to the pure jump case, that is associated to the measure $\tilde{N}(ds, dx)$. We write $D^J$ and $\delta^J$. We have the following result:

**Lemma**

For any $n$, consider the set $\Theta_{\mathcal{I}, \epsilon}^{n, \ast} = \{(\theta_1, \ldots, \theta_n) \in \Theta_{\mathcal{I}, \epsilon}^n : \theta_i \neq \theta_j \text{ if } i \neq j\}$. Then, for any $g_k \in L^2(\Theta_{\infty, 0}^{k, \ast})$ for $k \geq 1$ and $\omega \in \Omega^J$ we have, a.s.,

$$I_k(g_k)(\omega) = \int_{\Theta_{\mathcal{I}, \epsilon}^{k, \ast}} g_k(\theta_1 \ldots, \theta_k) \tilde{N}(\omega, d\theta_1) \cdots \tilde{N}(\omega, d\theta_k).$$

The proof is based on the fact that both expressions coincide for simple functions and define bounded linear operators.
**Relationship between $D^J$, $\delta^J$, $\Psi$ and $\Phi$**

For a fixed $k \geq 0$, consider $F = I_k(g_k)$ with $g_k$ a symmetric function of $L^2(\Theta^{k,*}_{\infty,0})$. Then, $F$ belongs to $DomD^J \cap Dom\Psi$ and

$$D^J I_k(g_k) = \Psi I_k(g_k), \ n \otimes \mathbb{P} - \text{a.e.}$$

For fixed $k \geq 1$, consider $u_\theta = I_k(g_k(\cdot, \theta))$ where $g_k(\cdot, \cdot) \in L^2(\Theta^{k+1,*}_{\infty,0})$ is symmetric with respect to the first $k$ variables. Assume also $u \in Dom\Phi$. Then,

$$\Phi(u) = \delta^J(u), \ \mathbb{P} - \text{a.e.}$$
Relationship between the operators

Let $F \in L^2(\Omega^J)$. Then, $F \in \text{Dom}D^J \iff \Psi F \in L^2(\Theta_\infty,0 \times \Omega_J)$, and in this case

$$D^J F = \Psi F, \ \nu \otimes P - \text{a.e.}$$

Let $u \in L^2(\Theta_\infty,0 \times \Omega_J) \cap \text{Dom}\Phi$. Then $u \in \text{Dom}\delta^J \iff \Phi u \in L^2(\Omega^J)$, and in this case

$$\delta^J u = \Phi u, \ \mathbb{P} - \text{a.s.}$$
As an application of the previous results in the pure jump case we hereafter prove a CHO-type formula as an integral representation of random variables in $L^1(\Omega^J)$.

**Theorem**

Let $F \in L^1(\Omega^J)$ and assume $\Psi F \in L^1(\ThetaM0 \times \Omega^J)$. Then

$$F = \mathbb{E}(F) + \Phi(\mathbb{E}(\Psi_{t,X} F | \mathcal{F}_{t-})), \text{ a.s.}$$
Remark

Observe that under the conditions of the previous theorem we have

$$\psi_{s,x} E[F | \mathcal{F}_{\Theta_{t-}}] = E[\psi_{s,x} F | \mathcal{F}_{\Theta_{t-}}] 1_{[0,t)}, \nu \otimes \mathbb{P} - a.e.$$
EXAMPLE 1 I

Consider a pure jump additive process $L$. On one hand, for any $t$, we have the Lévy-Itô decomposition:

$$L_t = \Gamma_t + \int_0^t \int_{\{|x|>1\}} xN(ds, dx) + \int_0^t \int_{\{|x|\leq 1\}} x\tilde{N}(ds, dx).$$

Consider $L_T$. Assume $\mathbb{E}(|L_T|) < \infty$. Recall that this is equivalently to

$$\int_0^t \int_{|x|>1} |x|\nu(ds, dx) < \infty.$$

Then we can write

$$L_t = \Gamma_t + \int_0^t \int_{\{|x|>1\}} x\nu(ds, dx) + \int_0^t \int_{\mathbb{R}} x\tilde{N}(ds, dx).$$
**Example 1 II**

On the other hand, applying the CHO formula, we have

\[ \psi_{s,x}L_T = x \mathbb{1}_{[0,T]}(s) \text{ and } E(\psi_{s,x}L_T|\mathcal{F}_{s-}) = x \mathbb{1}_{[0,T]}(s). \]

So, the hypothesis \( E(|L_T|) < \infty \) is equivalent to

\[ \mathbb{E} \int_0^T \int_{\mathbb{R}} |\psi_{s,x}L_T| \nu(ds, dx) < \infty \]

and

\[ L_T = \mathbb{E}(L_T) + \int_0^T \int_{\mathbb{R}} x \tilde{N}(ds, dx). \]

Observe that this is coherent with the previous decomposition because

\[ \mathbb{E}(L_T) = \Gamma_T + \int_0^T \int_{\{|x|>1\}} x \nu(ds, dx). \]
Let $X := \{X_t, t \in [0, T]\}$ be a pure jump Lévy process with triplet $(\gamma_L t, 0, \nu_L t)$. Let $S_t := e^{X_t}$ be an asset price process. Let $Q$ be a risk-neutral measure. In order $e^{-rt} e^{X_t}$ be a $Q$–martingale we need to assume some restrictions on $\nu_L$ and $\gamma_L$:

$$\int_{|x| \geq 1} e^x \nu_L(dx) < \infty$$

and

$$\gamma_L = \int_{\mathbb{R}} (e^y - 1 - y \mathbb{1}_{\{|y|<1\}}) \nu(dy).$$
These conditions allow us to write without losing generality,

\[ X_t = x + (r - c_2)t + \int_0^t \int \int_\mathbb{R} y \tilde{N}(ds, dy), \]

where

\[ c_2 := \int \int_\mathbb{R} (e^y - 1 - y) \nu_L(dy) \]

and \( N \) is a Poisson random measure under \( \mathbb{Q} \).

According to the CHO formula if \( F = S_T \in L^1(\Omega) \) and
\[ \mathbb{E}_\mathbb{Q}[\Psi_s,x S_T | \mathcal{F}_{s-}] \in L^1(\Omega \times [0, T]) \]
we have

\[ S_T = \mathbb{E}_\mathbb{Q}(S_T) + \int \int \int_{\Theta_{T,0}} \mathbb{E}_\mathbb{Q}[\Psi_s,x S_T | \mathcal{F}_{s-}] \tilde{N}(ds, dx). \]
Observe that

\[ \Psi_{s,x} S_T(\omega) = S_T(e^x - 1), \; \ell \times \nu_L \times \mathbb{Q} - \text{a.s.}, \]

and this process belongs to \( L^1(\Omega \times \Theta_{\infty,0}) \) if and only if

\[ \int_{\mathbb{R}} |e^x - 1| \nu_L(dx) < \infty. \]

Then, in this case, we have

\[ S_T = \mathbb{E}_Q(S_T) + \int_{\Theta_{T,0}} e^{r(T-s)}(e^x - 1) S_s \tilde{N}(ds, dx). \]

So, this result covers Lévy processes with finite activity and Lévy processes with infinite activity but finite variation.
Consider a pure jump volatility modulated additive driven Volterra (VMAV) process $X$ defined as

$$X(t) = \int_0^t g(t, s)\sigma(s)\,dJ(s)$$

provided the integral is well defined. Here $J$ is a pure jump additive processes, $g$ is a deterministic function and $\sigma$ is a predictable process with respect the natural completed filtration of $J$. 

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Recall that using the Lévy-Itô representation $J$ can be written as

$$J(t) = \Gamma_t + \int_{\Theta_t,0-\Theta_t,1} x\tilde{N}(ds, dx) + \int_{\Theta_t,1} xN(ds, dx),$$

where $\Gamma$ is a continuous deterministic function that we assume of bounded variation in order to admit integration with respect to $d\Gamma$. 
For each $t$, $X_t$ is well defined if

$$ (H1) : \int_0^\infty |g(t, s)\sigma(s)| \, d\Gamma_s < \infty, $$

$$ (H2) : \int_{\Theta_{\infty, 0}} (1 \wedge (g(t, s)\sigma(s)x)^2) \nu(dx, ds) < \infty, $$

and

$$ (H3) : \int_{\Theta_{\infty, 0}} |g(t, s)\sigma(s)x[\mathbb{1}_{\{|g(t,s)\sigma(s)x| \leq 1\}} - \mathbb{1}_{\{|x| \leq 1\}}]| \nu(dx, ds) < \infty. $$
Hereafter we discuss the problem of developing an integration theory with respect to $X$ as integrator, i.e. to give a meaning to

$$\int_0^t Y(s) dX(s)$$

for a fixed $t$ and a suitable stochastic processes $Y$. 

**INTEGRATION WITH RESPECT PURE JUMP VOLATILITY MODULATED VOLTERRA PROCESSES**

- Exploiting the representation of $J$, an integration with respect to $X$ can be treated as the sum of integrals with respect to the corresponding components of $J$.
- It is enough to define integrals with respect to $\int_0^t g(t, s)\sigma(s)d\Gamma_s$, $\int_0^t \int_{|x|\leq 1} g(t, s)\sigma(s)x\tilde{N}(ds, dx)$ and $\int_0^t \int_{|x|>1} g(t, s)\sigma(s)xN(ds, dx)$.
- Under the assumptions that $\Gamma$ has finite variation and using the fact that $N$ on $[0, t] \times \{|x| > \delta\}$, for any $\delta > 0$, is a.s. a finite measure, the integration with respect to the first and third term presents no difficulties.
We have to discuss the second term, specifically the case when $J$ has infinite activity and the corresponding $X$ is not a semimartingale. In fact, if $X$ was a semimartingale, we could perform the integration in the Itô sense.

We can refer to [BBPV] for a discussion of the conditions on $g$ in order $X$ be or not a semimartingale.

In [BBPV], an integral with respect to a non semimartingale $X$ driven by a Lévy process by means of the Malliavin-Skorohod calculus is defined. Their technique is naturally constrained to an $L^2$ setting.
Within the framework presented in this paper, we can extend the definition proposed in [BBPV] to reach out for additive processes beyond the $L^2$ setting.

Assume the following hypothesis on $X$ and $Y$:

- For $s \geq 0$, the mapping $t \mapsto g(t, s)$ is of bounded variation on any interval $[u, v] \subseteq (s, \infty)$.
- The function

$$\mathcal{K}_g(Y)(t, s) := Y(s)g(t, s) + \int_s^t (Y(u) - Y(s))g(du, s), \quad t > s,$$

is well defined a.s., in the sense that $(Y(u) - Y(s))$ is integrable with respect to $g(du, s)$ as a pathwise Lebesgue-Stieltjes integral.
Integration with respect pure jump volatility modulated Volterra processes

- The mappings

\[
(s, x) \rightarrow \mathcal{K}_g(Y)(t, s)\sigma(s)x \mathbb{1}_{\Theta_{t,0} - \Theta_{t,1}}(s, x)
\]

and

\[
(s, x) \rightarrow \psi_{s,x}(\mathcal{K}_g(Y)(t, s)\sigma(s))x \mathbb{1}_{\Theta_{t,0} - \Theta_{t,1}}(s, x)
\]

belong to \(\text{Dom}\Phi\).
Integration with respect pure jump volatility modulated Volterra processes

Then, the following integral, is well defined:

\[
\int_0^t Y(s) d\left( \int_0^s \int_{|x| \leq 1} g(s, u) \sigma(u) x \tilde{N}(du, dx) \right)
\]

\[
:= \Phi(x \mathcal{K}_g(Y)(t, s) \sigma(s) \mathbb{1}_{\Theta_{t,0} - \Theta_{t,1}}(s, x))
\]

\[
+ \Phi(x \psi_{s,x}(\mathcal{K}_g(Y)(t, s)) \sigma(s) \mathbb{1}_{\Theta_{t,0} - \Theta_{t,1}}(s, x))
\]

\[
+ \mathcal{E}(x \psi_{s,x}(\mathcal{K}_g(Y)(t, s)) \sigma(s) \mathbb{1}_{\Theta_{t,0} - \Theta_{t,1}}(s, x)).
\]
Integration with respect pure jump volatility modulated Volterra processes

- This result extends the definition in [BBPV] to any pure jump additive process \( J \), i.e. beyond square integrability.
- The proof relies on the definitions of \( \Phi \), \( \Psi \) and the developed calculus rules.
- In the finite activity case,
  \[
  L^2(\Theta_\infty, 0 \times \Omega^J) \subseteq L^1(\Theta_\infty, 0 \times \Omega^J)
  \]
  and our result is an extension of the definition in [BBPV].
- In the infinite activity case, our Theorem covers cases not covered by [BBPV] and viceversa.
Hereafter we give a classical example of a pure jump Lévy process without second moment as a driver and we consider a kernel function $g$ of shift type, i.e. it only depends on the difference $t - s$. For simplicity we assume moreover $\sigma \equiv 1$. The chosen kernel appears in applications to turbulence.

Assume $L$ to be a symmetric $\alpha$–stable Lévy process, for $\alpha \in (0, 2)$. It corresponds to the triplet $(0, 0, \nu_L)$ with $\nu_L(dx) = c|x|^{-1-\alpha}dx$. 
Example II

Take

\[ g(t, s) := (t - s)^{\beta - 1} e^{-\lambda (t - s)} 1_{[0,t]}(s) \]

with \( \beta \in (0, 1) \) and \( \lambda > 0 \). Note that

\[ g(du, s) = -g(u, s) \left( \frac{1 - \beta}{u - s} + \lambda \right) du. \]
**Example III**

We concentrate on the component

$$J(t) = \int_0^t \int_{\{|x|\leq 1\}} x\tilde{N}(ds, dx),$$

and so first of all on the definition of the integral

$$X(t) := \int_0^t g(t, s)dJ(s) = \int_0^t \int_{|x|\leq 1} g(t, s)x\tilde{N}(ds, dx), \quad t \geq 0.$$
Example IV

In relation with this integral, that is not a semimartingale, we have four situations:

1. If $\alpha \in (0, 1)$ and $\beta > \frac{1}{2}$, $g(t, s)x$ belongs to $L^1 \cap L^2$.
2. If $\alpha \in (0, 1)$ and $\beta \leq \frac{1}{2}$, $g(t, s)x$ belongs to $L^1$ but not to $L^2$.
3. If $\alpha \in [1, 2)$ and $\beta > \frac{1}{2}$, $g(t, s)x$ belongs to $L^2$ but not to $L^1$.
4. If $\alpha \in [1, 2)$ and $\beta \leq \frac{1}{2}$, $g(t, s)x$ belongs not to $L^2$ nor to $L^1$.

Only in case (4) the integral is not necessarily well defined.
**Example V**

Just to show the types of computation involved, let us consider the particular case of a VMAV process as integrand. Namely,

\[
Y(s) = \int_0^s \int_{|x| \leq 1} \phi(s - u)x\tilde{N}(du, dx), \quad 0 \leq s \leq T,
\]

where \( \phi \) is a positive continuous function such that the integral \( Y \) is well defined.

Consider the case \( \alpha < 1 \) and \( \beta \in (0, 1) \), not covered by [BBPV] if \( \beta \leq \frac{1}{2} \).
**Example VI**

In order to see that \( \int_0^t Y(s-)dX(s) \) is well defined we have to check:

1. The process \((Y(u) - Y(s))\) is integrable with respect to \(g(du, s)\) on \((s, t]\), as a pathwise Lebesgue-Stieltjes integral.

2. The mappings

\[
(s, x) \mapsto x \mathcal{K}_g(Y)(t, s) \mathbb{1}_{[0,t]}(s) \mathbb{1}_{\{|x| \leq 1\}}
\]

and

\[
(s, x) \mapsto x \psi_{s,x}(\mathcal{K}_g(Y)(t, s)) \mathbb{1}_{[0,t]}(s) \mathbb{1}_{\{|x| \leq 1\}}
\]

belong to \(Dom\Phi\).
We have

\[ \mathcal{K}_g(Y)(t, s) = g(t, s) \int_{[0,s]} \int_{|x| \leq 1} \phi(s - v) x \tilde{N}(dv, dx) \]

\[ - \int_s^t g(u, s) \left( \frac{1 - \beta}{u - s} + \lambda \right) \int_{[s,u]} \int_{|x| \leq 1} \phi(u - v) x \tilde{N}(dv, dx) du \]

\[ - \int_s^t g(u, s) \left( \frac{1 - \beta}{u - s} + \lambda \right) \int_{[0,s]} \int_{|x| \leq 1} \left[ \phi(u - v) - \phi(s - v) \right] x \tilde{N}(dv, dx) du \]
**Example VIII**

In terms of \( \Phi \) we can rewrite

\[
\begin{align*}
\mathcal{K}_g(Y)(t, s) &= g(t, s)\Phi(\phi(s - \cdot))1_{\{|x| \leq 1\}}1_{[0,s)} \\
&- \int_s^t g(u, s)\left(\frac{1 - \beta}{u - s} + \lambda\right)\Phi(\phi(u - \cdot))1_{\{|x| \leq 1\}}1_{[s,u)}(\cdot)\,du \\
&- \int_s^t g(u, s)\left(\frac{1 - \beta}{u - s} + \lambda\right)\Phi([\phi(u - \cdot) - \phi(s - \cdot)])1_{\{|x| \leq 1\}}1_{[0,s)}(\cdot)\,du.
\end{align*}
\]

Moreover we have

\[
\begin{align*}
\Psi_{s,x}\mathcal{K}_g(Y)(t, s) &= -x1_{\{|x| \leq 1\}} \int_s^t g(u, s)\phi(u - s)\left(\frac{1 - \beta}{u - s} + \lambda\right)1_{[0,u)}(s)\,du.
\end{align*}
\]
So, it is enough to check that the mappings

\[(s, x) \rightarrow x \mathcal{K}_g(Y)(t, s) \mathbb{1}_{[0, t]}(s) \mathbb{1}_{\{|x| \leq 1\}}\]

and

\[(s, x) \rightarrow x \psi_{s, x}(\mathcal{K}_g(Y)(t, s)) \mathbb{1}_{[0, t]}(s) \mathbb{1}_{\{|x| \leq 1\}}\]

are in \(L^1(\Theta_{\infty, 0} \times \Omega)\).
If for example we consider the case $\phi(y) = y^\gamma$ with $\gamma > 0$ and $\beta + \gamma \geq 1$ is not difficult to check the mappings are in $L^1$ and we conclude that the integral

$$\int_0^t Y(s-)dX(s)$$

is well defined.
Thank you for the attention

Tak

Gràcies